



# A Short Note on the Derivation of the Atmospheric Transfer Function for a Communications Channel and its Connection to Associated Propagation Parameters

Robert M. Manning  
Glenn Research Center, Cleveland, Ohio

## The NASA STI Program Office . . . in Profile

Since its founding, NASA has been dedicated to the advancement of aeronautics and space science. The NASA Scientific and Technical Information (STI) Program Office plays a key part in helping NASA maintain this important role.

The NASA STI Program Office is operated by Langley Research Center, the Lead Center for NASA's scientific and technical information. The NASA STI Program Office provides access to the NASA STI Database, the largest collection of aeronautical and space science STI in the world. The Program Office is also NASA's institutional mechanism for disseminating the results of its research and development activities. These results are published by NASA in the NASA STI Report Series, which includes the following report types:

- **TECHNICAL PUBLICATION.** Reports of completed research or a major significant phase of research that present the results of NASA programs and include extensive data or theoretical analysis. Includes compilations of significant scientific and technical data and information deemed to be of continuing reference value. NASA's counterpart of peer-reviewed formal professional papers but has less stringent limitations on manuscript length and extent of graphic presentations.
- **TECHNICAL MEMORANDUM.** Scientific and technical findings that are preliminary or of specialized interest, e.g., quick release reports, working papers, and bibliographies that contain minimal annotation. Does not contain extensive analysis.
- **CONTRACTOR REPORT.** Scientific and technical findings by NASA-sponsored contractors and grantees.

- **CONFERENCE PUBLICATION.** Collected papers from scientific and technical conferences, symposia, seminars, or other meetings sponsored or cosponsored by NASA.
- **SPECIAL PUBLICATION.** Scientific, technical, or historical information from NASA programs, projects, and missions, often concerned with subjects having substantial public interest.
- **TECHNICAL TRANSLATION.** English-language translations of foreign scientific and technical material pertinent to NASA's mission.

Specialized services that complement the STI Program Office's diverse offerings include creating custom thesauri, building customized databases, organizing and publishing research results . . . even providing videos.

For more information about the NASA STI Program Office, see the following:

- Access the NASA STI Program Home Page at <http://www.sti.nasa.gov>
- E-mail your question via the Internet to [help@sti.nasa.gov](mailto:help@sti.nasa.gov)
- Fax your question to the NASA Access Help Desk at 301-621-0134
- Telephone the NASA Access Help Desk at 301-621-0390
- Write to:  
NASA Access Help Desk  
NASA Center for Aerospace Information  
7121 Standard Drive  
Hanover, MD 21076



# A Short Note on the Derivation of the Atmospheric Transfer Function for a Communications Channel and its Connection to Associated Propagation Parameters

Robert M. Manning  
Glenn Research Center, Cleveland, Ohio

National Aeronautics and  
Space Administration

Glenn Research Center

Available from

NASA Center for Aerospace Information  
7121 Standard Drive  
Hanover, MD 21076

National Technical Information Service  
5285 Port Royal Road  
Springfield, VA 22100

Available electronically at <http://gltrs.grc.nasa.gov>

# **A Short Note on the Derivation of the Atmospheric Transfer Function for a Communications Channel and its Connection to Associated Propagation Parameters**

Robert M. Manning  
National Aeronautics and Space Administration  
Glenn Research Center  
Cleveland, Ohio 44135

## **Abstract**

The systems engineering description of a wideband communications channel is provided which is based upon the fundamental propagation aspects of the problem. In particular, the well known time variant description of a channel is formulated from the basic multiple scattering processes that occur in a random propagation medium. Such a connection is required if optimal processing methods are to be applied to mitigate the deleterious random fading and ‘multipathing’ of the channel. An example is given which demonstrates how the effective bandwidth of the channel is diminished due to atmospheric propagation impairments.

## **Introduction**

From the communications engineering standpoint, the intervening propagation channel between a transmitter and receiver is treated as a separate system-element whereby only the input and output characteristics of the channel are defined; the intricate structure and mechanism within the system-element communications channel is not identified nor considered (see, for example, [Rummler, 1979]). The associated atmospheric transfer function is traditionally empirically determined. Such an approach has found applicability in the design and specification of a communications system that must operate in the deleterious channel mechanism of the earth’s atmosphere (see, for example, [Proakis, 1989]). However, in the case of wideband transmission through planetary atmospheres which can display a host of different propagation mechanisms, one needs to add a descriptive level to the channel which reflects these mechanisms. Thus, one must go back to fundamentals and ‘re-derive’ the system engineering description of a communications channel from basic propagation principles. From such a detailed description can one effectively specify and design optimal processing algorithms to mitigate the prevailing multipath and fading of the propagation channel. This goal forms the subject matter of this work.

## **Fundamental Propagation Aspects of the Problem**

The relevant starting point is, of, course, the Maxwell equations; with a spatially and temporally dependent random permittivity  $\varepsilon = \varepsilon(\vec{r}, t)$ , one has for the electric field  $\vec{E} = \vec{E}(\vec{r}, t)$  and magnetic field  $\vec{B} = \vec{B}(\vec{r}, t)$  of the modulated wave field

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t} \quad (1a)$$

$$\nabla \times \vec{H} = \frac{1}{c} \frac{\partial (\epsilon \vec{E})}{\partial t} \quad (1b)$$

$$\nabla \cdot (\epsilon \vec{E}) = 0 \quad (1c)$$

which are, necessarily, also random quantities. In the usual way, operating on eq. (1a) with the operator  $\nabla \times$  and employing the vector identity  $\nabla \times \nabla \times \vec{E} = -\nabla^2 \vec{E} + \nabla(\nabla \cdot \vec{E})$  gives

$$\begin{aligned} \nabla^2 \vec{E} - \nabla(\nabla \cdot \vec{E}) &= \frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \vec{H}) \\ &= \left(\frac{1}{c}\right)^2 \frac{\partial^2}{\partial t^2} (\epsilon \vec{E}) \end{aligned} \quad (2)$$

where eq. (1b) was used in the second line. Equation (1c) is now expanded to yield

$$\nabla \cdot \vec{E} = \vec{E} \cdot \nabla(\ln \epsilon) \quad (3)$$

and upon using this result in eq. (2) finally gives

$$\nabla^2 \vec{E} - \left(\frac{1}{c}\right)^2 \frac{\partial^2}{\partial t^2} (\epsilon \vec{E}) = \nabla(\vec{E} \cdot \nabla(\ln \epsilon)) \quad (4)$$

which is a stochastic vector wave equation due to the random source term on the right side.

Equation (4) is a general relation describing the evolution of the vector electric field propagating through the random permittivity field  $\epsilon$ . The term on the right side of this equation describes the depolarization of the electric field as it propagates thus making eq. (4) a vector equation for the three components of the electric field [Manning, 1993, p. 8]. Although a general solution can be obtained for this equation, this circumstance complicates the subsequent analysis using this approach and a simplification is possible whereby the depolarization term is neglected. The necessary conditions under which this is valid have been established and can be found elsewhere [Manning, 1993, appendix A]. So as not to deter from the present analysis, this depolarization term will be neglected in what is to follow.

Thus, with these considerations prevailing, the neglect to the depolarization term allows eq. (4) to decouple into three scalar equations, one for each component to the  $E$  field. Therefore, one is now concerned with the stochastic scalar wave equation

$$\nabla^2 E - \left(\frac{1}{c}\right)^2 \frac{\partial^2}{\partial t^2} (\epsilon E) = 0 \quad (5)$$

for one of the components  $E$  of the electric field. At this point, it is important to remember that the permittivity field  $\varepsilon(\vec{r}, t)$  is a random function of position  $\vec{r}$  and time  $t$  so this equation does not have any straightforward solution and must be treated only with statistical methods; with the  $E$  field being a random field, due to the presence of the  $\varepsilon$  function, the  $E$  field can only be described in terms of its statistics, i.e., its mean, variance, etc., in space and time. Departing from the usual “narrow-band” treatment (see appendix A) of eq. (5) by which it is reduced to the Helmholtz equation, the field  $E$  will be taken to contain a range of spectral frequency components due to modulation of the propagating wave field; thus, eq. (5) must be dealt with in its entirety.

In what is to follow, the function  $\varepsilon$  will be written as a sum of a deterministic part  $\varepsilon_0 = \varepsilon_0(\vec{r}, t) \equiv 1$  and a random part  $\tilde{\varepsilon} = \tilde{\varepsilon}(\vec{r}, t)$  with the associated conditions  $|\tilde{\varepsilon}| \ll 1$  and  $\langle \varepsilon \rangle = 0$ . (Here, the notation  $\langle y \rangle$  indicates the ensemble average of a random quantity  $y$ .) Making this assignment in eq. (5) and simplifying gives

$$\nabla^2 E - \left(\frac{1}{c}\right)^2 \frac{\partial^2}{\partial t^2} E = \left(\frac{1}{c}\right)^2 \frac{\partial^2}{\partial t^2} (\tilde{\varepsilon} E) \quad (6)$$

Using the temporal Fourier Transform

$$E(\vec{r}, \omega) = \int_{-\infty}^{\infty} E(\vec{r}, t) \exp(i\omega t) dt \quad (7)$$

and applying it to eq. (6) gives

$$\nabla^2 E(\vec{r}, \omega) - \left(\frac{1}{c}\right)^2 \int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2} E(\vec{r}, t) \exp(i\omega t) dt = \left(\frac{1}{c}\right)^2 \int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2} [\tilde{\varepsilon}(\vec{r}, t) E(\vec{r}, t)] \exp(i\omega t) dt \quad (8)$$

The second term on the left side of this equation becomes, after two integrations by parts with applicable boundary conditions at  $t \rightarrow \pm\infty$ ,

$$\left(\frac{1}{c}\right)^2 \int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2} E(\vec{r}, t) \exp(i\omega t) dt = -k^2 E(\vec{r}, \omega) \quad (9)$$

Similarly, the member on the right side becomes

$$\left(\frac{1}{c}\right)^2 \int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2} [\tilde{\varepsilon}(\vec{r}, t) E(\vec{r}, t)] \exp(i\omega t) dt = -k^2 \int_{-\infty}^{\infty} \tilde{\varepsilon}(\vec{r}, t) E(\vec{r}, t) \exp(i\omega t) dt \quad (10)$$

Equation (8) can then be written in the temporal frequency domain as

$$\nabla^2 E(\vec{r}, \omega) + k^2 E(\vec{r}, \omega) = -k^2 \int_{-\infty}^{\infty} \tilde{\varepsilon}(\vec{r}, t) E(\vec{r}, t) \exp(i\omega t) dt \quad (11)$$

The exact relation given by eq. (11) can be simplified by using the quasi-stationary approximation (see appendix A) whereby, relative to the inherent temporal variations of the functions  $E(\vec{r}, t)$  and  $\exp(i\omega t)$  that contribute to the integrand of eq. (11), the temporal variations of  $\epsilon(\vec{r}, t)$  are ‘slow’ (see eq. (A.3) of appendix A). Thus, one can approximately factor  $\epsilon(\vec{r}, t)$  out of the integrand on the right side of eq. (11) and obtain the simpler relation

$$\nabla^2 E(\vec{r}, \omega) + k^2 E(\vec{r}, \omega) = -k^2 \tilde{\epsilon}(\vec{r}, t) E(\vec{r}, \omega) \quad (12)$$

which is isomorphic to that used in investigations of the propagation of monochromatic radiation.

Any of the well-known methods of analysis in stochastic wave propagation theory can now be used to study the ramifications of this relation [Manning, 1993], i.e., the Rytov method, the parabolic equation method, or many of the methods developed for quantum field theory. In order to maintain generality in attempting to relate the rigorous methods of wave propagation theory with the *ad hoc* methods employed in communications theory, the widely applicable quantum theoretic methods will be used. In particular, use will be made of a factorization technique which will concatenate within a single parameter all orders of multiple scattering of the wave field within the random medium, resulting in what is known as the Lippmann-Schwinger equation; the single parameter is known in quantum field theory as the  $T$ -matrix but, for purposes here will be called the random scattering operator  $T$ . With this in hand, it will be shown that the resulting relations can indeed be put into the form of time-variant systems theory which is so widely used in multipath propagation but, more importantly, giving explicit meaning to the quantities of the time-variant systems model.

To this end, one writes the solution of eq. (12) as (see appendix B)

$$E(\vec{r}, \omega) = E_0(\vec{r} - \vec{R}, \omega) + \int_{-\infty}^{\infty} G_0(\vec{r}, \vec{r}') q(\vec{r}', \omega) d^3 r' \quad (13)$$

where  $E_0(\vec{r} - \vec{R}, \omega)$  is the initial field distribution due to a source at  $\vec{R}$  of frequency  $\omega$ ,

$$q(\vec{r}', \omega) \equiv -k^2 \tilde{\epsilon}(\vec{r}', t) E(\vec{r}', \omega) \quad (14)$$

is the scattered field source term and the Green function  $G_0(\vec{r}, \vec{r}')$  is the solution of the related differential equation

$$\nabla^2 G_0(\vec{r}, \vec{r}') + k^2 G_0(\vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}') \quad (15)$$

which is

$$G_o(\vec{r}, \vec{r}') = G_o(|\vec{r} - \vec{r}'|) = -\frac{1}{4\pi} \frac{\exp(ik|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|} \quad (16)$$

Thus, remembering that  $k \equiv \omega/c$  which allows eq. (16) to be stated in terms of  $\omega$ , the solution for eq. (12) becomes

$$E(\vec{r}, \omega) = E_0(\vec{r} - \vec{R}, \omega) + \frac{k^2}{4\pi} \int_{-\infty}^{\infty} \frac{\exp\left(i\frac{\omega}{c}|\vec{r} - \vec{r}'|\right)}{|\vec{r} - \vec{r}'|} \tilde{\epsilon}(\vec{r}', t) E(\vec{r}', \omega) d^3 r' \quad (17)$$

Hence, even though the propagation of modulated wave fields is considered in this development, due the fact that the spectral content of the wave field is significantly smaller than the frequency of the carrier wave, and due to the quasi-stationary approximation, the form of the solution of the randomly scattered field is identical to that of the monochromatic case. Although eq. (17) will be developed further in what is to follow, a brief digression will be made concerning the meaning of the content of eq. (17) over that of the monochromatic case which will form the foundation of the analysis of digital transmission.

Returning to the time-domain using the Fourier transform relationship inverse to that of eq. (7), viz.,

$$E(\vec{r}, t) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} E(\vec{r}, \omega) \exp(-i\omega t) d\omega \quad (18)$$

and applying this to eq. (17) yields

$$\begin{aligned} E(\vec{r}, t) &= E_0(\vec{r} - \vec{R}, t) + \frac{k^2}{4\pi} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\left(i\frac{\omega}{c}|\vec{r} - \vec{r}'| - i\omega t\right)}{|\vec{r} - \vec{r}'|} \tilde{\epsilon}(\vec{r}', t) E(\vec{r}', \omega) d^3 r' d\omega \\ &= E_0(\vec{r} - \vec{R}, t) + \frac{k^2}{4\pi} \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-i\omega \left[t - \frac{|\vec{r} - \vec{r}'|}{c}\right]\right) E(\vec{r}', \omega) d\omega \right\} \frac{\tilde{\epsilon}(\vec{r}', t)}{|\vec{r} - \vec{r}'|} d^3 r' \\ &= E_0(\vec{r} - \vec{R}, t) + \frac{k^2}{4\pi} \int_{-\infty}^{\infty} E\left(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c}\right) \frac{\tilde{\epsilon}(\vec{r}', t)}{|\vec{r} - \vec{r}'|} d^3 r' \end{aligned} \quad (19)$$

In the Fraunhofer diffraction region, in which the receiving point is sufficiently removed from the scattering position, one can expand the quantity  $|\vec{r} - \vec{r}'|$ , viz.,

$$|\vec{r} - \vec{r}'| \approx r - \hat{n} \cdot \vec{r}' + \dots$$

where  $\hat{n} \equiv \vec{r}/r$  is the unit vector directed from the center of the scattering volume to the receiver point thus allowing eq. (19) to be written as

$$E(\vec{r}, t) = E_0(\vec{r} - \vec{R}, t) + \frac{k^2}{4\pi r} \int_V \tilde{\epsilon}(\vec{r}', t) E\left(\vec{r}', t - \frac{r}{c} + \frac{\hat{n} \cdot \vec{r}'}{c}\right) d^3 r' \quad (20)$$

where  $V$  is the volume comprising the scattering region. Finally, iterating once the solution of eq. (20) yields the Born approximation for the single scattering events, i.e.,

$$E(\vec{r}, t) = E_0(\vec{r} - \vec{R}, t) + \frac{k^2}{4\pi r} \int_v \tilde{\epsilon}(\vec{r}', t) E_0\left(\vec{r}' - \vec{R}, t - \frac{r}{c} + \frac{\hat{n} \cdot \vec{r}'}{c}\right) d^3 r' \quad (21)$$

## Multiple Scattering Within the Channel

The form of the solution in eq. (21) shows how the resulting total field  $E(\vec{r}, t)$  is the sum of a direct, unscattered field, i.e.,  $E_0(\vec{r} - \vec{R}, t)$ , and a scattered component which is a function of the time delay  $t_d = r/c - \hat{n} \cdot \vec{r}'/c$ . Such multipath phenomena results in a temporal spread induced into the communications channel; see, e.g., [Proakis, 1989]. In addition, the perturbation introduced by the random permittivity  $\tilde{\epsilon}(\vec{r}, t)$  will give rise to a fading channel.

In fact, if one were to continually iterate the solution given by eq. (20), a multiple scattering series is obtained whereby the  $n^{\text{th}}$  term in the series describes  $n$  multiple scattering events. Such multiple scattering gives rise to a total received signal which is a sum of signals that each traverse separate paths through the propagation medium resulting in multipath interference. This phenomena serves to degrade the available bandwidth of the communications channel. This will be shown in a much more explicit form in what is to follow.

Returning now to eq. (17) with the goal of obtaining a relationship that includes all orders of multiple scattering, one commences with the straightforward iterative solution of eq. (17). At the outset, however, it will be beneficial to simplify the notation somewhat. Defining the 'propagator' function

$$P(\vec{r}, \vec{r}', \omega) \equiv \frac{k^2}{4\pi} \frac{\exp\left[i\frac{\omega}{c}|\vec{r} - \vec{r}'|\right]}{|\vec{r} - \vec{r}'|} \quad (22)$$

allows eq. (17) to be rewritten as

$$E(\vec{r}, \omega) = E_0(\vec{r} - \vec{R}, \omega) + \int_{-\infty}^{\infty} P(\vec{r}, \vec{r}', \omega) \tilde{\epsilon}(\vec{r}', t) E(\vec{r}', \omega) d^3 r' \quad (23)$$

As is shown in appendix C, one can obtain an iterative solution to eq. (23) which, upon factorization, can be written as

$$E(\vec{r}, \omega) = E_0(\vec{r} - \vec{R}, \omega) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\vec{r}, \vec{r}', \omega) T(\vec{r}', \vec{r}'', t, \omega) E_0(\vec{r}'' - \vec{R}, \omega) d^3 r'' d^3 r' \quad (24)$$

where the factor

$$T(\vec{r}', \vec{r}'', t, \omega) \equiv \tilde{\epsilon}(\vec{r}', t) \left[ \delta(\vec{r}' - \vec{r}'') + P(\vec{r}', \vec{r}'', \omega) \tilde{\epsilon}(\vec{r}'', t) + \int_{-\infty}^{\infty} P(\vec{r}', \vec{r}''', \omega) \tilde{\epsilon}(\vec{r}''', t) P(\vec{r}'', \vec{r}'', \omega) \tilde{\epsilon}(\vec{r}'', t) d^3 r''' + \dots \right] \quad (25)$$

is the scattering operator which convolves all possible multiple scattering events that occur between the scattering inhomogeneities  $\tilde{\epsilon}$ .

Equation (24) is of the form of the well-known Lippmann-Schwinger equation of quantum electrodynamic scattering theory [Lippman and Schwinger, 1950], [Goldberger and Watson, 1964]. As mentioned above, the utility of this relation rests in the explicit connection of the initial, unperturbed field  $E_0$  with that of the total scattered random field  $E$  through the use of the random scattering operator  $T$  convolving all the scattering mechanisms within the random media through which the wave propagates.

Equations (24) and (25) can also be written in a convenient but less explicit operator notation, i.e.,

$$E = E_0 + PTE_0, \quad T = \epsilon(1 + P\epsilon + P\epsilon P\epsilon + \dots) \quad (26)$$

where the '1' in the expression for the scattering operator is the integral identity  $\delta(\vec{r} - \vec{r}')$ . The operator representation for  $T$  suggests a further factorization allowing it to be written as the recursive operator relationship

$$T = \epsilon(1 + PT) \quad (27)$$

This is indeed the case as can be established by using the same factorization method as employed to derive eq. (24) which gives in coordinate representation

$$T(\vec{r}, \vec{r}', t, \omega) = \tilde{\epsilon}(\vec{r}, t) \left[ \delta(\vec{r} - \vec{r}') + \int_{-\infty}^{\infty} P(\vec{r}, \vec{r}'', \omega) T(\vec{r}', \vec{r}'', t, \omega) d^3 r'' \right] \quad (28)$$

## The Time-Variant Systems Description of the Channel

Equation (24) is of a form that easily lends itself to the establishment of a time-variant systems description of propagation through a random medium. To this end, one lets the initial field distribution be given as a temporally modulated wave

$$E_0(\vec{r}) = A(\vec{r})m(t)\exp(-i\omega_0 t) \quad (29)$$

where  $A(\vec{r})$  is the initial spatial distribution of the wave field,  $m(t)$  is the modulating signal, and  $\omega_0$  is the carrier frequency. Using the Fourier transform relationship of eq. (7) gives

$$E_0(\vec{r}, \omega) = A(\vec{r})M(\omega - \omega_0) \quad (30)$$

where

$$M(\omega - \omega_0) \equiv \int_{-\infty}^{\infty} m(t)\exp[i(\omega - \omega_0)t]dt \quad (31)$$

is the frequency spectrum of the modulation. Substituting eq. (30) into eq. (24) yields

$$E(\vec{r}, \omega) = M(\omega - \omega_0) V(\vec{r} - \vec{R}, \omega, t) \quad (32)$$

where

$$V(\vec{r} - \vec{R}, \omega, t) = A(\vec{r} - \vec{R}) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\vec{r}, \vec{r}', \omega) T(\vec{r}', \vec{r}'', \omega, t) A(\vec{r}'' - \vec{R}) d^3 r'' d^3 r' \quad (33)$$

Inverse transforming eq. (32) back to the temporal coordinate gives

$$E(\vec{r}, t) = \left( \frac{1}{2\pi} \right) \int_{-\infty}^{\infty} M(\nu) V(\vec{r} - \vec{R}, \nu + \omega_0, t) \exp(-i\nu t) d\nu \exp(-i\omega_0 t) \quad (34)$$

Defining the envelope  $e(\vec{r}, t)$  of the field by the relation  $E(\vec{r}, t) = e(\vec{r}, t) \exp(-i\omega_0 t)$  gives

$$e(\vec{r}, t) = \left( \frac{1}{2\pi} \right) \int_{-\infty}^{\infty} M(\nu) V(\vec{r} - \vec{R}, \nu + \omega_0, t) \exp(-i\nu t) d\nu \quad (35)$$

Using the relation

$$M(\nu) = \int_{-\infty}^{\infty} m(t) \exp(i\nu t) dt \quad (36)$$

in eq. (35) and rearranging terms allows one to finally write for the envelope of the modulated propagating wave field

$$e(\vec{r}, t) = \int_{-\infty}^{\infty} m(t') h(\vec{r} - \vec{R}, t - t', t) dt' \quad (37)$$

where the impulse response of the atmospheric communications channel is given by

$$h(\vec{r} - \vec{R}, t - t', t) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} V(\vec{r} - \vec{R}, \nu + \omega_0, t) \exp[-i\nu(t - t')] d\nu \quad (38)$$

From this impulse response function, one can define correlation as well as power spectral density functions from which one can define several characteristics of the fading multipath communications channel [Proakis, 1989]. For example, one can form expressions for the delay power spectrum, the spaced-frequency, spaced-time correlation function, as well as the Doppler power spectrum of the channel. However, the most direct demonstration of the effects of the multipath and fading on the modulation attendant on the channel is through the response function for the average power.

One can obtain the response function for the average power as follows. It is important to note that the quantity  $V(\vec{r}-\vec{R}, \nu+\omega_0, t)$  is a random function since the scattering operator which enters into its definition (eq. (33)) is a random function of the permittivity  $\tilde{\epsilon}(\vec{r}, t)$  given by eq. (25). Thus, any variable or function related to it will also be a random quantity. Thus, one can only refer to the measurable aspects of these quantities in terms of their statistical descriptions such as the average and higher order statistical moments. For this reason, one can only calculate the average power from the use of eq. (35), viz,

$$I(\vec{r}, t) \equiv \langle e(\vec{r}, t) e^*(\vec{r}, t) \rangle \\ = \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} M(\nu) M^*(\nu') \langle V(\vec{r}-\vec{R}, \nu+\omega_0, t) V^*(\vec{r}-\vec{R}, \nu'+\omega_0, t) \rangle \\ \cdot \exp[-i(\nu-\nu')t] d\nu d\nu' \quad (39)$$

Defining the two-frequency mutual coherence function (MCF) by

$$\Gamma(\vec{r}-\vec{R}, \nu+\omega_0, \nu'+\omega_0, t) \equiv \langle V(\vec{r}-\vec{R}, \nu+\omega_0, t) V^*(\vec{r}-\vec{R}, \nu'+\omega_0, t) \rangle \quad (40)$$

as well as the center and difference frequencies, respectively,

$$\nu_c \equiv \frac{\nu+\nu'}{2}, \quad \nu_d \equiv \nu-\nu' \quad (41)$$

allows eq. (39) to be written as

$$I(\vec{r}, t) = \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} M\left(\nu_c + \frac{1}{2}\nu_d\right) M^*\left(\nu_c - \frac{1}{2}\nu_d\right) \Gamma(\vec{r}-\vec{R}, \nu_c+\omega_0, \nu_d, t) \exp[-i\nu_d t] d\nu_c d\nu_d \quad (42)$$

where the MCF is taken also to be a function of the center and difference frequencies of  $\nu+\omega_0$  and  $\nu'+\omega_0$ .

This general relation can be simplified if the spectral densities of the modulation are such that they fall-off as a function of  $\nu_c$  for values of  $\nu_c \ll \omega_0$  thus essentially yielding a negligible contribution to the integral at values of  $\nu_c \sim \omega_0$ . In this instance, one can write  $\Gamma(\vec{r}-\vec{R}, \nu_c+\omega_0, \nu_d, t) \approx \Gamma(\vec{r}-\vec{R}, \omega_0, \nu_d, t)$ , i.e., the MCF becomes approximately independent of  $\nu_c$  and, thus, eq. (42) takes the form

$$I(\vec{r}, t) = \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \Gamma_M\left(\frac{\nu_d}{2}\right) \Gamma(\vec{r}-\vec{R}, \omega_0, \nu_d, t) \exp[-i\nu_d t] d\nu_d \quad (43)$$

where

$$\Gamma_M\left(\frac{\omega}{2}\right) \equiv \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} M\left(v_c + \frac{1}{2}\omega\right) M^*\left(v_c - \frac{1}{2}\omega\right) dv_c \quad (44)$$

is the input modulation power spectrum and the MCF  $\Gamma(\vec{r} - \vec{R}, \omega_0, v_d, t)$  acts as a time-variant transfer function.

Proceeding further in the simplification of this result, one can apply a quasi-steady state assumption based on the fact that the channel characteristics, ultimately connected to the time varying permittivity  $\epsilon(\vec{r}, t)$ , slowly evolve over the time scale of digital transmission along the link. In this event, the MCF can be made independent of time allowing one to write  $\Gamma(\vec{r} - \vec{R}, \omega_0, v_d, t) \approx \Gamma(\vec{r} - \vec{R}, \omega_0, v_d)$ . Equation (43) then becomes

$$I(\vec{r}, t) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} \Gamma_M\left(\frac{v_d}{2}\right) \Gamma(\vec{r} - \vec{R}, \omega_0, v_d) \exp[-iv_d t] dv_d \quad (45)$$

Fourier transforming this relation with respect to  $\omega$  yields

$$I(\vec{r}, \omega) = \Gamma_M\left(\frac{\omega}{2}\right) \Gamma(\vec{r} - \vec{R}, \omega_0, \omega) \quad (46)$$

This shows how, in the case for a time-independent channel in which the MCF is a function of only the difference frequency, the modulation power spectrum  $\Gamma_M(\omega/2)$  is modified by the MCF

### A Simple Time-Invariant Example

An illustrative example will now be given which shows how, due to the deleterious multipath effects concatenated in the time-independent MCF  $\Gamma(\vec{r}, \omega_0, \omega/2)$ , the effective bandwidth of the time-invariant communications channel is degraded. Consider the propagation of a Gaussian pulse given by

$$m(t) = \exp\left(-\frac{t^2}{T_0^2}\right) \quad (47)$$

From eq. (36), the associated spectrum of the pulse is

$$M(v) = \sqrt{\pi} T_0 \exp\left(-\frac{T_0^2}{4} v^2\right) \quad (48)$$

The bandwidth  $\Delta v$  associated with this pulse spectrum can be operationally defined by  $\exp(-1) = \exp(-T_0^2(\Delta v)^2/4)$ ; thus

$$\Delta v = \frac{2}{T_0} \quad (49)$$

The input signal power spectrum corresponding to eq. (48) is, from eq. (44),

$$\Gamma_M\left(\frac{\omega}{2}\right) = \sqrt{\frac{\pi}{2}} T_0 \exp\left(-\frac{T_0^2}{4} \omega^2\right) \quad (50)$$

Now suppose, for ease of calculation, that the atmospheric MCF of the channel is that of a plane wave (i.e., independent of spatial position) also of Gaussian form, i.e.,

$$\Gamma\left(\vec{r}, \omega_0, \frac{\omega}{2}\right) = \Gamma\left(\omega_0, \frac{\omega}{2}\right) = \exp(-\beta^2(\omega_0) \omega^2) \quad (51)$$

where the coefficient  $\beta(\omega_0)$  is a function of the propagation parameters that characterize the propagation path, viz., number and type of aerosols, amount of atmospheric turbulence, etc. Using eqs. (50) and (51) in eq. (46) and performing the inverse Fourier Transform gives

$$\begin{aligned} I(\vec{r}, t) &= \int_{-\infty}^{\infty} \Gamma(\vec{r} - \vec{R}, \omega_0, \omega) \Gamma_M\left(\frac{\omega}{2}\right) \exp(-i\omega t) d\omega \\ &= \frac{T_0}{\sqrt{T_0^2 + 8\beta^2}} \exp\left(-\frac{2t^2}{T_0^2 + 8\beta^2}\right) \end{aligned} \quad (52)$$

The situation with  $\beta(\omega_0) = 0$  corresponds to the ‘vacuum’ case in which the effects of the intervening atmosphere are absent. In this instance, eq. (52) gives for the vacuum case

$$I_0(\vec{r}, t) \equiv I(\vec{r}, t)|_{\beta=0} = \frac{1}{T_0} \exp\left(-\frac{2t^2}{T_0^2}\right) \quad (53)$$

Comparison of these two expressions yields the fact that the effective bandwidth of the communications channel is decreased from that given in eq. (49) to that given by

$$\Delta v = \frac{2}{\sqrt{T_0^2 + 8\beta^2}} \quad (54)$$

## Summary

The usual time-variant systems theory approach to the design and analysis of communications systems that must operate through a random channel, such as that of the atmosphere, has been derived from first principles using multiple scattering theory. In particular, the atmospheric transfer function of the communications channel is directly related to the

multiple scattering equations of stochastic propagation theory. From this, the two-frequency mutual coherence function can be defined by considering the average power transmitted within a modulated wave field. It was then shown how, in the simple time-invariant case, this two-frequency coherence function modifies the modulation power spectrum within the random channel and results in a degraded channel bandwidth.

## Appendix A

At the outset of all ‘narrow-band’ analyses of the stochastic Maxwell equations used to describe statistical propagation situations[Manning, 1993, pp. 2-7, and references therein], one assumes a wave field  $E = E(\vec{r}, t)$  of single temporal frequency  $\omega$  and decomposes it using the relation

$$E(\vec{r}, t) = \mathbb{E}(\vec{r}, t) \exp(-i\omega t) \quad (\text{A.1})$$

allowing eq. (5) to be written as

$$\nabla^2 \mathbb{E} + k^2 \varepsilon \mathbb{E} = -\frac{2ik}{c} \frac{\partial(\varepsilon \mathbb{E})}{\partial t} + \frac{1}{c^2} \frac{\partial^2(\varepsilon \mathbb{E})}{\partial t^2} \quad (\text{A.2})$$

where  $k \equiv \omega/c$  is the free space wave number. In this expression, the wave field  $\mathbb{E} = \mathbb{E}(\vec{r}, t)$  is still a random function of time but is devoid of the temporal oscillatory component. The only temporal variation enters in through the random permittivity function  $\varepsilon(\vec{r}, t)$ . One writes this function as a sum  $\varepsilon(\vec{r}, t) = \varepsilon_0(\vec{r}, t) + \tilde{\varepsilon}(\vec{r}, t)$  of a deterministic part  $\varepsilon_0(\vec{r}, t)$  and a random part  $\tilde{\varepsilon}(\vec{r}, t)$ , and for simplicity, lets  $\varepsilon_0(\vec{r}, t) = 1$  and assumes  $|\varepsilon(\vec{r}, t)| \ll 1$ . Using this subsequent relation in eq. (A.2) and analyzing the partial temporal derivatives that appear shows that they can be deemed negligible so long as the condition

$$\frac{\partial \tilde{\varepsilon}}{\partial t} \ll \frac{1}{\mathbb{E}} \left( \frac{\partial \mathbb{E}}{\partial t} \right) \quad (\text{A.3})$$

prevails. In this instance, eq. (A.2) simplifies to

$$\nabla^2 \mathbb{E} + k^2 \varepsilon \mathbb{E} = -k \tilde{\varepsilon} \mathbb{E} \quad (\text{A.4})$$

i.e., a stochastic Helmholtz wave equation which is only parametric in time.. The condition of eq. (A.3), which states that the level of the temporal variation of the permittivity field is much smaller than that which is characteristic of the electric field of the propagation wave, is thus known as the *quasi-static approximation*. The structure of eq. (A.4) is such that the random function  $\varepsilon$ , which can be statistically specified depending upon the propagation scenario, multiplies the function being sought, i.e., the stochastic field  $\mathbb{E}$ . This circumstance is the source of difficulty in obtaining solutions to eq. (A.4) and forms the subject of the many investigations on the subject in the past 40 years. The approach taken in the present work assumes an arbitrary range for the frequency content of the wave field necessitating the retention of the temporal derivatives in eq. (5). It is important to remember that in eq. (5), both the functions  $\varepsilon$  and  $E$  are functions of time and thus the indicated operations contain several derivative terms which, in the present case, must each be examined for their level of contribution.

## Appendix B

To easily show this development, one writes, for notational clarity, eq. (12) in operator notation, viz.,

$$\hat{L}E(\vec{r}, \omega) = q(\vec{r}, \omega), \quad \hat{L} \equiv \nabla^2 + k^2 \quad (\text{B.1})$$

and assumes a solution of this differential equation to be of the form

$$E(\vec{r}, \omega) = E_0(\vec{r} - \vec{R}, \omega) + \int G_0(\vec{r}, \vec{r}') q(\vec{r}', \omega) d^3 r' \quad (\text{B.2})$$

where  $E_0(\vec{r} - \vec{R}, \omega)$  is the initial (unscattered) field distribution and the kernel  $G_0(\vec{r}, \vec{r}')$  of the integrand must be determined from the structure of the problem given in eq. (B.1). Thus, applying the operator  $\hat{L}$  to eq. (B.2) and using eq. (B.1) gives

$$\begin{aligned} \hat{L}E(\vec{r}, \omega) &= \hat{L}E_0(\vec{r} - \vec{R}, \omega) + \hat{L} \int G_0(\vec{r}, \vec{r}') q(\vec{r}', \omega) d^3 r' \\ &= \int \hat{L}G_0(\vec{r}, \vec{r}') q(\vec{r}', \omega) d^3 r' \\ &= q(\vec{r}, \omega) \end{aligned} \quad (\text{B.3})$$

where, by definition, one has  $\hat{L}E_0(\vec{r} - \vec{R}, \omega) = 0$ , and for the last two lines of this development to hold, one must necessarily have

$$\hat{L}G_0(\vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}') \quad (\text{B.4})$$

which is eq. (15). The solution of eq. (B.4) can be obtained using Fourier Transform methods.

## Appendix C

Iterating successive expressions of eq. (23) gives (suppressing the  $t$  and  $\omega$  dependence for simplicity)

$$\begin{aligned}
 E(\vec{r}) = & E_0(\vec{r} - \vec{R}) + \int_{-\infty}^{\infty} P(\vec{r}, \vec{r}') \tilde{\epsilon}(\vec{r}') E_0(\vec{r}' - \vec{R}) d^3 r' + \\
 & + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\vec{r}, \vec{r}') \tilde{\epsilon}(\vec{r}') P(\vec{r}', \vec{r}'') \tilde{\epsilon}(\vec{r}'') E_0(\vec{r}'' - \vec{R}) d^3 r' d^3 r'' + \\
 & + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\vec{r}, \vec{r}') \tilde{\epsilon}(\vec{r}') P(\vec{r}', \vec{r}''') \tilde{\epsilon}(\vec{r}''') P(\vec{r}''', \vec{r}''') \tilde{\epsilon}(\vec{r}''') E_0(\vec{r}'' - \vec{R}) d^3 r' d^3 r'' d^3 r''' + \dots
 \end{aligned} \tag{C.1}$$

The basic idea here is to factor out appropriate quantities within each of the integrals within each term on the right side so as to isolate those factors that exclusively describe the multiple scattering process. To this end, one can immediately factor an integral and a propagator function allowing eq. (C.1) to be written

$$\begin{aligned}
 E(\vec{r}) = & E_0(\vec{r} - \vec{R}) + \int_{-\infty}^{\infty} P(\vec{r}, \vec{r}') \left\{ \tilde{\epsilon}(\vec{r}') E_0(\vec{r}' - \vec{R}) + \right. \\
 & + \int_{-\infty}^{\infty} P(\vec{r}', \vec{r}'') \tilde{\epsilon}(\vec{r}'') E_0(\vec{r}'' - \vec{R}) d^3 r'' + \\
 & \left. + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\vec{r}', \vec{r}''') \tilde{\epsilon}(\vec{r}''') P(\vec{r}''', \vec{r}''') \tilde{\epsilon}(\vec{r}''') E_0(\vec{r}'' - \vec{R}) d^3 r'' d^3 r''' + \dots \right\} d^3 r'
 \end{aligned} \tag{C.2}$$

One now needs to factor out the initial field distribution  $E_0(\vec{r} - \vec{R})$ ; it is thus required to isolate each of the factors  $E_0(\vec{r} - \vec{R})$  in each member of eq. (C.2) so they can be factored from the remaining portions of the integrands; the latter will then form a series in multiple scattering of the wave field within the random medium. Making use of the fact that the dummy variables in each of the integrands can be interchanged with one another as well as, from eq. (22),  $P(\vec{r}, \vec{r}') = P(\vec{r}', \vec{r})$ , the fourth member of the right side of eq. (C.2) can be rewritten as

$$\begin{aligned}
 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\vec{r}', \vec{r}''') \tilde{\epsilon}(\vec{r}''') P(\vec{r}''', \vec{r}''') \tilde{\epsilon}(\vec{r}''') E_0(\vec{r}'' - \vec{R}) d^3 r'' d^3 r''' = \\
 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\vec{r}', \vec{r}''') \tilde{\epsilon}(\vec{r}''') P(\vec{r}''', \vec{r}''') \tilde{\epsilon}(\vec{r}''') E_0(\vec{r}'' - \vec{R}) d^3 r'' d^3 r'''
 \end{aligned}$$

where the transcription  $\vec{r}'' \leftrightarrow \vec{r}'''$  has been applied. Thus, the former factor  $E_0(\vec{r}''' - \vec{R})$  has been changed to  $E_0(\vec{r}'' - \vec{R})$  without changing the result of the double integral; this process can, in principle, be carried out for all remaining terms that appear in the series. It is very important to note that this process can only be employed to convert all  $E_0(\vec{r}^{(n)} - \vec{R})$  factors to  $E_0(\vec{r}'' - \vec{R})$ . This is so because if each such term were converted to  $E_0(\vec{r}' - \vec{R})$ , i.e., with a single prime on the

$\vec{r}$  coordinate, the third member on the right of eq. (C.2) would not yield to the process due to the lack of enough dummy variables. Thus, attention must now be turned to the second member on the right side to convert the  $E_0(\vec{r}' - \vec{R})$  factor to one involving the coordinate  $\vec{r}''$ , i.e.,  $E_0(\vec{r}'' - \vec{R})$ . One can simply use the prescription  $E_0(\vec{r}' - \vec{R}) = \delta(\vec{r}' - \vec{r}'')E_0(\vec{r}'' - \vec{R})$  noting that the integral over the  $\vec{r}''$  coordinate will return the proper function. With these considerations in mind,  $E_0(\vec{r}'' - \vec{R})$  factors can be taken out of each term along with the integral over  $\vec{r}''$ , and eq. (C.2) finally becomes

$$E(\vec{r}) = E_0(\vec{r} - \vec{R}) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\vec{r}, \vec{r}') T(\vec{r}', \vec{r}'') E_0(\vec{r}'' - \vec{R}) d^3 r'' d^3 r' \quad (C.3)$$

where

$$T(\vec{r}', \vec{r}'') \equiv \tilde{\epsilon}(\vec{r}') \left[ \delta(\vec{r}' - \vec{r}'') + P(\vec{r}', \vec{r}'') \tilde{\epsilon}(\vec{r}'') + \int_{-\infty}^{\infty} P(\vec{r}', \vec{r}''') \tilde{\epsilon}(\vec{r}''') P(\vec{r}''', \vec{r}'') \tilde{\epsilon}(\vec{r}'') d^3 r''' + \dots \right] \quad (C.4)$$

## References

Goldberger, M. L., and K. M. Watson (1964), *Collision Theory* (Wiley, New York), p.754

Lippman, B. A., and J. Schwinger (1950), “Variational Principles for Scattering Processes”, *Phys. Rev.* **79**, pp. 469-480.

Manning, R. M. (1993), *Stochastic Electromagnetic Image Propagation*, (McGraw Hill, New York).

Proakis, J. G. (1989), *Digital Communications*, 2<sup>nd</sup> Ed. (McGraw-Hill, New York). Chap. 7.

Rummler, W. D. (1979), “A New Selective Fading Model: Application to Propagation Data”, *Bell Syst. Tech. J.* **58**, pp. 1037-1071.

**REPORT DOCUMENTATION PAGE**Form Approved  
OMB No. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

<b>1. AGENCY USE ONLY (Leave blank)</b>		<b>2. REPORT DATE</b> March 2004	<b>3. REPORT TYPE AND DATES COVERED</b> Technical Memorandum	
<b>4. TITLE AND SUBTITLE</b> A Short Note on the Derivation of the Atmospheric Transfer Function for a Communications Channel and its Connection to Associated Propagation Parameters			<b>5. FUNDING NUMBERS</b>  WBS-22-302-20-2F	
<b>6. AUTHOR(S)</b>  Robert M. Manning				
<b>7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)</b>  National Aeronautics and Space Administration John H. Glenn Research Center at Lewis Field Cleveland, Ohio 44135-3191			<b>8. PERFORMING ORGANIZATION REPORT NUMBER</b>  E-14451	
<b>9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)</b>  National Aeronautics and Space Administration Washington, DC 20546-0001			<b>10. SPONSORING/MONITORING AGENCY REPORT NUMBER</b>  NASA TM-2004-212975	
<b>11. SUPPLEMENTARY NOTES</b>  Responsible person, Robert M. Manning, organization code 5640, 216-433-6750.				
<b>12a. DISTRIBUTION/AVAILABILITY STATEMENT</b>  Unclassified - Unlimited Subject Categories: 17, 32, and 36 Available electronically at <a href="http://gltrs.grc.nasa.gov">http://gltrs.grc.nasa.gov</a> This publication is available from the NASA Center for AeroSpace Information, 301-621-0390.			<b>12b. DISTRIBUTION CODE</b>	
<b>13. ABSTRACT (Maximum 200 words)</b>  The systems engineering description of a wideband communications channel is provided which is based upon the fundamental propagation aspects of the problem. In particular, the well known time variant description of a channel is formulated from the basic multiple scattering processes that occur in a random propagation medium. Such a connection is required if optimal processing methods are to be applied to mitigate the deleterious random fading and 'multipathing' of the channel. An example is given which demonstrates how the effective bandwidth of the channel is diminished due to atmospheric propagation impairments.				
<b>14. SUBJECT TERMS</b>  Electromagnetic wave transmission; Wave propagation optical communication			<b>15. NUMBER OF PAGES</b> 23	
			<b>16. PRICE CODE</b>	
<b>17. SECURITY CLASSIFICATION OF REPORT</b> Unclassified	<b>18. SECURITY CLASSIFICATION OF THIS PAGE</b> Unclassified	<b>19. SECURITY CLASSIFICATION OF ABSTRACT</b> Unclassified	<b>20. LIMITATION OF ABSTRACT</b>	



