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# Impact of Functionally Graded Cylinders: Theory

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# Impact of Functionally Graded Cylinders: Theory

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**Abstract.** This final report summarizes the work funded under the Grant **NAG3-2411** during the 04/05/2000-04/04/2001 period. The objective of this one-year project was to generalize the theoretical framework of the two-dimensional higher-order theory for the analysis of cylindrical functionally graded materials/structural components employed in advanced aircraft engines developed under past NASA-Glenn funding. The completed generalization significantly broadens the theory's range of applicability through the incorporation of dynamic impact loading capability into its framework. Thus it makes possible the assessment of the effect of damage due to fuel impurities, or the presence of submicron-level debris, on the life of functionally graded structural components. Applications involving advanced turbine blades and structural components for the reusable-launch vehicle (RLV) currently under development will benefit from the completed work. The theory's predictive capability is demonstrated through a numerical simulation of a one-dimensional wave propagation set up by an impulse load in a layered half-plane. Full benefit of the completed generalization of the higher-order theory described in this report will be realized upon the development of a related computer code.

## 1. INTRODUCTION

Functionally graded materials (FGMs) are a new generation of composites wherein the microstructural details are spatially varied through nonuniform distribution of the reinforcement phase(s), by using reinforcement with different properties, sizes and shapes, as well as by interchanging the roles of reinforcement and matrix phases in a continuous manner. The result is a microstructure that produces continuously changing thermal and mechanical properties at the macroscopic or continuum level. This new concept of engineering the material's microstructure allows, for the first time, to fully integrate both the material and structural considerations into the final design of structural components.

Most computational strategies for the response of FGMs do not explicitly couple the material's heterogeneous microstructure with the structural global analysis. Rather, local effective or macroscopic properties are first obtained through homogenization based on a chosen micromechanics scheme, and then used in a global thermomechanical analysis. This often leads to erroneous results when the temperature gradient is large with respect to the dimension of the inclusion phase, the characteristic dimension of the inclusion phase is large relative to the global dimensions of the composite, and the number of uniformly or nonuniformly distributed inclusions is relatively small (Ref.

[1]). As a result of the limitations of the uncoupled approach, a new higher-order micromechanical theory for FGMs (HOTFGM), that explicitly couples the local and global effects, has been developed in the Cartesian coordinate system for applications involving rectangular plate-like structural components under NASA funding (Refs. [2-19]). The development of the theory has been justified by comparison with the results obtained using the standard micromechanics approach which neglects the micro-macrostructural coupling effects (Refs. [4,7]). Summaries of significant results and accomplishments generated using this theory, and the utility of functionally graded microstructures in enhancing the performance of plate-like structural components subjected to through-thickness thermal gradients have been outlined in Refs. [9,18,19,20].

In order to exploit the already-proven predictive capabilities of HOTFGM to the fullest in meeting the challenges and needs of the aerospace and aircraft engine industries for a greater number of stronger, lighter and more durable structural components, new versions of HOTFGM have been recently developed for applications involving cylindrical bodies of revolution. The generalization of HOTFGM to problems involving cylindrical geometries makes possible the analysis, optimization and design of functionally graded structural components, such as rotor disks, combustor linings and blisk blades, for use in advanced aircraft engines. Thus far, two versions of the cylindrical higher-order theory have been developed. The quasi one-dimensional version enables analysis, design and optimization of cylindrical bodies of revolution subjected to axisymmetric thermomechanical loading that are reinforced by either continuous or discontinuous fibers with variable spacing in the radial direction (Ref. [21]). The recently completed development of the two-dimensional version, which also admits the presence of cooling channels, enables analysis, design and optimization of cylindrical bodies of revolution with functionally graded microstructures in the radial and circumferential directions (Ref. [22]). The thermomechanical loading involves arbitrary distribution of surface tractions and temperatures applied to the boundaries of fully or partially enclosed cylindrical bodies of revolution in the plane that contains the functionally graded microstructure. This significant generalization provides analysis and design capabilities for a wider range of structural components employed in advanced aircraft engines.

The utility of the developed Cartesian and cylindrical versions of the higher-order theory has been demonstrated through applications to the following technologically important problems:

- Investigation of the effect of microstructure on thermal and stress fields in MMC plates and cylinders
- Investigation of the use of functionally graded architectures in reducing edge effects in MMC plates
- Optimization of functionally graded microstructures in MMC plates and cylinders
- Development of guidelines for the design of special coatings in exhaust nozzle applications under NASA/Pratt & Whitney Space Act Agreement
- Investigation of the microstructural effects in functionally graded TBCs
- Effect of bond coat interfacial roughness and oxide film thickness on the inelastic response of plasma-sprayed TBCs
- Effect of graded bond coats on the inelastic response of plasma-sprayed TBCs

While the recently completed cylindrical higher-order theory contains transient thermal loading capability, the mechanical loading capability does not include dynamic effects appearing in the governing force equilibrium differential equations. This, in turn, excludes the possibility of gaging the effect of impact loading (by fuel impurities or submicron debris, for instance) on damage evolution in such applications as advanced, functionally graded thermal barrier coatings, for instance. The objective of the work summarized in this report, therefore, was to extend the two-dimensional higher-order theory for cylindrical functionally graded structural components by incorporating dynamic impact loading capability. This extension funded under the Grant **NAG3-2411** during the 04/05/2000-04/04/2001 period makes possible the assessment of the effect of foreign object impact on the potential for damage evolution in advanced turbine blade coatings. It complements current nation-wide efforts by a number of government agencies to develop a new generation of turbine blade coatings capable of operating longer in low-cost fuel environments containing different types of impurities. As the impact problem also plays an important role in a number of other technologically important applications that are important to the nation's security interests (graded body armour, for instance), the completed work significantly broadens the range of technologically important applications of the cylindrical higher-order theory. However, the completed work is limited to the development of the theoretical framework that enables modeling of impact-induced wave propagation in the radial and circumferential directions of functionally graded cylindrical structural components. Therefore, the description of this theoretical development forms the major part of this report. Full utilization of the impact-loading capability requires the development of the related computer code and its validation. These tasks remain to be completed under future funding. To demonstrate the potential of the developed theoretical framework, a small computer code was developed to simulate one-dimensional wave propagation due to impulse loading in a layered half-plane as a special case of the general two-dimensional theory, and the numerical predictions were validated through comparison with an exact analytical solution. These results are described at the end of the report.

## 2. MODEL DESCRIPTION

The present FGM theory is based on the geometric model of a heterogeneous composite occupying the region  $R_0 \leq r \leq R_1$ ,  $0 \leq \theta \leq \Theta$ ,  $|z| < \infty$ , where  $r, \theta, z$  are cylindrical coordinates, see Fig. 1. The composite is reinforced in the  $r - \theta$  plane by an arbitrary distribution of infinitely long fibers oriented along the axial  $z$ -axis, or by finite-length inclusions that are arranged in a periodic manner in the axial direction. The microstructure of the heterogeneous composite is discretized into  $N_p$  and  $N_q$  cells in the intervals  $R_0 \leq r \leq R_1$  and  $0 \leq \theta \leq \Theta$ , respectively. As in the Cartesian version of the higher-order theory (see Ref. [19], for instance), the generic cell  $(p, q, s)$  used to construct the composite consists of eight subcells designated by the triplet  $(\alpha\beta\gamma)$ , where each index  $\alpha, \beta, \gamma$  takes on the value 1 or 2 to indicate the relative position of the given subcell along the  $r-$ ,  $\theta-$ , and  $z-$  axis, respectively. The indices  $p$  and  $q$ , whose ranges are  $p = 1, 2, \dots, N_p$  and  $q = 1, 2, \dots, N_q$ , identify the generic cell in the  $r - \theta$  plane and thus remain constant along the axial  $z$  axis. For the axial direction, the corresponding index  $s$  having an infinite range is introduced. The dimensions of the generic cell along the periodic axial direction,  $l_1, l_2$ , are fixed for the given configuration; whereas the dimensions along the  $r-$  and  $\theta-$  axes or the FG directions,  $d_1^{(p)}, d_2^{(p)}$ , and  $\theta_1^{(q)}, \theta_2^{(q)}$ , can vary in an arbitrary fashion such that  $D = \sum_{p=1}^{N_p} (d_1^{(p)} + d_2^{(p)})$  and  $\Theta = \sum_{q=1}^{N_q} (\theta_1^{(q)} + \theta_2^{(q)})$ .

Given the applied thermomechanical loading in the  $(r - \theta)$ -plane, an approximate solution for

the temperature and the time-dependent displacement fields is constructed based on the volumetric averaging of the field equations together with the imposition of boundary and continuity conditions in the average sense between the subvolumes used to characterize the material's microstructure. This is accomplished by approximating the temperature field in each subcell of the generic cell using a quadratic expansion in the local coordinates  $(\bar{r}^{(\alpha)}, \bar{\theta}^{(\beta)}, \bar{z}^{(\gamma)})$  centered at the subcell's center. Similarly, the time-dependent displacement field in the FG direction in each subcell is approximated using a quadratic expansion in local coordinates within the subcell. The displacement field in the periodic axial direction, on the other hand, is approximated using linear expansion in local coordinates to reflect the periodic character of the composite's microstructure along the  $z$ -axis. A higher order representation of the temperature and displacement fields is necessary to capture the local effects created by the thermomechanical field gradients, the microstructure of the composite, and the finite dimensions in the FG direction.

The unknown coefficients associated with each term in the temperature field expansion are then obtained by constructing a system of equations that satisfies the requirements that the steady state heat equation is satisfied in a volumetric sense, and the thermal and heat flux continuity conditions within a given cell, as well as between a given cell and adjacent cells, are imposed in an average sense, together with the applied boundary conditions.

Due to the presence of the inertia effects in the governing mechanical equations, on the other hand, the second time derivatives of the unknown time-dependent coefficients associated with each term in the displacement field expansion are obtained in this case by constructing a system of equations that satisfies the requirements that the elastodynamic equations are satisfied in a volumetric sense, and the displacement and traction vectors continuity conditions within a given cell, as well as between a given cell and adjacent cells, are imposed in an average sense, together with the applied time-dependent boundary conditions. This system of second order ordinary differential equations is solved in a stepwise manner in time by employing an approximate explicit scheme. Once these coefficients have been determined at the current time  $t$ , all field variables can be readily established. This procedure is continued until the specified final time is reached.

### 3. THERMAL ANALYSIS

Let the composite be subjected to the steady-state temperature distributions  $T_T(\theta)$  on the top surface ( $r = R_1$ ),  $T_B(\theta)$  on the bottom surface ( $r = R_0$ ),  $T_L(r)$  on the left surface ( $\theta = 0$ ), and  $T_R(r)$  on the right surface ( $\theta = \Theta$ ). Under these circumstances, the heat flux field in the material occupying the subcell  $(\alpha\beta\gamma)$  of the  $(p, q, s)$ th cell must satisfy the steady state heat equation in cylindrical coordinates  $(r, \theta, z)$ . This equation is given by

$$\frac{\partial q_r^{(\alpha\beta\gamma)}}{\partial \bar{r}^{(\alpha)}} + \frac{1}{R^{(\alpha\beta\gamma)} + \bar{r}^{(\alpha)}} q_r^{(\alpha\beta\gamma)} + \frac{\partial q_\theta^{(\alpha\beta\gamma)}}{\partial \bar{y}^{(\beta)}} + \frac{\partial q_z^{(\alpha\beta\gamma)}}{\partial \bar{z}^{(\gamma)}} = 0 \quad (1)$$

where  $R^{(\alpha\beta\gamma)}$  is the distance of the  $(\alpha\beta\gamma)$  subcell's center from the origin,  $\bar{y}^{(\beta)} = R^{(\alpha\beta\gamma)}\bar{\theta}^{(\beta)}$ , and  $q_i^{(\alpha\beta\gamma)}$  ( $i = r, \theta, z$ ) are the components of the heat flux vector in the subcell. These components are derived from the temperature  $T^{(\alpha\beta\gamma)}$  in the subcell according to the Fourier law:

$$q_r^{(\alpha\beta\gamma)} = -k_r^{(\alpha\beta\gamma)} \frac{\partial T^{(\alpha\beta\gamma)}}{\partial \bar{r}^{(\alpha)}} \quad (2)$$

$$q_{\theta}^{(\alpha\beta\gamma)} = -k_{\theta}^{(\alpha\beta\gamma)} \frac{\partial T^{(\alpha\beta\gamma)}}{\partial \bar{y}^{(\beta)}} \quad (3)$$

$$q_z^{(\alpha\beta\gamma)} = -k_z^{(\alpha\beta\gamma)} \frac{\partial T^{(\alpha\beta\gamma)}}{\partial \bar{z}^{(\gamma)}} \quad (4)$$

where  $k_i^{(\alpha\beta\gamma)}$  denote the thermal conductivities of the material in the subcell.

As in the analysis of the steady-state heat equation in Cartesian coordinates, the temperature field in the subcell is expanded quadratically in terms of the local coordinates  $(\bar{r}^{(\alpha)}, \bar{y}^{(\beta)}, \bar{z}^{(\gamma)})$  as follows

$$\begin{aligned} T^{(\alpha\beta\gamma)} &= T_{(000)}^{(\alpha\beta\gamma)} + \bar{r}^{(\alpha)} T_{(100)}^{(\alpha\beta\gamma)} + \bar{y}^{(\beta)} T_{(010)}^{(\alpha\beta\gamma)} \\ &+ \frac{1}{2} (3\bar{r}^{(\alpha)2} - \frac{d_{\alpha}^{(p)2}}{4}) T_{(200)}^{(\alpha\beta\gamma)} + \frac{1}{2} (3\bar{y}^{(\beta)2} - \frac{h_{\beta}^{(q)2}}{4}) T_{(020)}^{(\alpha\beta\gamma)} + \frac{1}{2} (3\bar{z}^{(\gamma)2} - \frac{l_{\gamma}^2}{4}) T_{(002)}^{(\alpha\beta\gamma)} \end{aligned} \quad (5)$$

where  $h_{\beta}^{(q)} = R^{(\alpha\beta\gamma)} \theta_{\beta}^{(q)}$ . The microvariables  $T_{(000)}^{(\alpha\beta\gamma)}$ , which is the volume-averaged temperature within the subcell, and  $T_{(lmn)}^{(\alpha\beta\gamma)}$  ( $l, m, n = 0, 1$ , or  $2$  with  $l + m + n \leq 2$ ) are unknown coefficients that are determined in the manner described below. It should be noted that the temperature expansion given in eqn (5) does not contain a linear term in the local coordinates  $\bar{z}^{(\gamma)}$ . This follows directly from the assumed periodicity in the axial  $z$ -direction and symmetry with respect to the lines  $\bar{z}^{(\gamma)} = 0$  for  $\gamma = 1$  and  $2$ .

Given the six unknown quantities associated with each subcell (i.e.  $T_{(000)}^{(\alpha\beta\gamma)}, \dots, T_{(002)}^{(\alpha\beta\gamma)}$ ) and eight subcells within each generic cell,  $48N_p N_q$  unknown quantities must be determined for a composite with  $N_p$  and  $N_q$  subcells containing arbitrary specified materials. These quantities are determined by first satisfying the heat conduction equation, as well as the first and second moment of this equation in each subcell, in a volumetric sense in view of the above temperature field approximation. Subsequently, continuity of heat flux and temperature is imposed in an average sense at the interfaces separating adjacent subcells as well as neighboring cells. Fulfillment of these field equations and continuity conditions together with the imposed thermal boundary conditions at the top, bottom, left and right surfaces of the composite provides the necessary  $48N_p N_q$  equations for the  $48N_p N_q$  unknown coefficients in the temperature field expansion. We begin the outline of steps to generate the required  $48N_p N_q$  equations by first considering an arbitrary  $(p, q, s)$ th cell in the interior of the composite (i.e.,  $p = 2, \dots, N_p - 1$  and  $q = 2, \dots, N_q - 1$ ). This produces  $48(N_p - 2)(N_q - 2)$  equations. The additional equations are obtained by considering the boundary cells (i.e.,  $p = 1, N_p$  and  $q = 1, N_q$ ). For these cells, most of the preceding relations also hold, with the exception of some of the interfacial continuity conditions between adjacent cells that are replaced by the specified boundary conditions.

In the course of satisfying the steady-state heat equation in a volumetric sense, it is convenient to define the following flux quantities:

$$[Q_{i(l,m,n)}^{(\alpha\beta\gamma)}]^{(p,q,s)} = \frac{1}{v_{(\alpha\beta\gamma)}^{(p,q,s)}} \int_{-d_{\alpha}^{(p)}/2}^{d_{\alpha}^{(p)}/2} \int_{-h_{\beta}^{(q)}/2}^{h_{\beta}^{(q)}/2} \int_{-l_{\gamma}/2}^{l_{\gamma}/2} (\bar{r}^{(\alpha)})^l (\bar{y}^{(\beta)})^m (\bar{z}^{(\gamma)})^n q_i^{(\alpha\beta\gamma)} d\bar{r}^{(\alpha)} d\bar{y}^{(\beta)} d\bar{z}^{(\gamma)} \quad (6)$$

where  $i = r, \theta, z$ ;  $l, m, n = 0, 1$ , or  $2$  with  $l + m + n \leq 2$  and  $v_{(\alpha\beta\gamma)}^{(p,q,s)} = d_{\alpha}^{(p)} h_{\beta}^{(q)} l_{\gamma}$  being the volume of the subcell. For  $l = m = n = 0$ ,  $Q_{i(0,0,0)}^{(\alpha\beta\gamma)}$  is the average value of the heat flux component  $q_i^{(\alpha\beta\gamma)}$

in the subcell, whereas for other values of  $(l, m, n)$  this equation defines higher order heat fluxes. These flux quantities can be evaluated explicitly in terms of the coefficients  $T_{(lmn)}^{(\alpha\beta\gamma)}$  by performing the required volume integration. This yields the following nonvanishing zeroth- and first-order heat fluxes in terms of the unknown coefficients in the temperature field expansion

$$Q_{r(0,0,0)}^{(\alpha\beta\gamma)} = -k_r^{(\alpha\beta\gamma)} T_{(100)}^{(\alpha\beta\gamma)} \quad (7)$$

$$Q_{\theta(0,0,0)}^{(\alpha\beta\gamma)} = -k_\theta^{(\alpha\beta\gamma)} T_{(010)}^{(\alpha\beta\gamma)} \quad (8)$$

$$Q_{r(1,0,0)}^{(\alpha\beta\gamma)} = -k_r^{(\alpha\beta\gamma)} \frac{d_\alpha^{(p)2}}{4} T_{(200)}^{(\alpha\beta\gamma)} \quad (9)$$

$$Q_{\theta(0,1,0)}^{(\alpha\beta\gamma)} = -k_\theta^{(\alpha\beta\gamma)} \frac{h_\beta^{(q)2}}{4} T_{(020)}^{(\alpha\beta\gamma)} \quad (10)$$

$$Q_{z(0,0,1)}^{(\alpha\beta\gamma)} = -k_z^{(\alpha\beta\gamma)} \frac{l_\gamma^2}{4} T_{(002)}^{(\alpha\beta\gamma)} \quad (11)$$

Satisfaction of the zeroth, first, and second moment of the steady-state heat equation (1) results into the following eight relationships among the first-order heat fluxes  $Q_{i(l,m,n)}^{(\alpha\beta\gamma)}$  in the different  $(\alpha\beta\gamma)$  subcells of the  $(p, q, s)$ th cell, after some involved algebraic manipulations (see Ref. [12] for a complete derivation in Cartesian coordinates)

$$\left[ \frac{12}{d_\alpha^2} Q_{r(1,0,0)}^{(\alpha\beta\gamma)} + \frac{12}{h_\beta^2} Q_{\theta(0,1,0)}^{(\alpha\beta\gamma)} + \frac{12}{l_\gamma^2} Q_{z(0,0,1)}^{(\alpha\beta\gamma)} + \frac{1}{R^{(\alpha\beta\gamma)}} Q_{r(0,0,0)}^{(\alpha\beta\gamma)} \right]^{(p,q,s)} = 0 \quad (12)$$

where the triplet  $(\alpha\beta\gamma)$  assumes all permutations of the integers 1 and 2.

The continuity of the heat fluxes at the subcell interfaces and between individual cells in the radial direction, imposed in an average sense, is ensured by

$$\begin{aligned} \left[ \frac{12}{d_1^2} Q_{r(1,0,0)}^{(1\beta\gamma)} \right]^{(p,q,s)} &= -Z_1 \left[ \frac{12}{d_2^2} Q_{r(1,0,0)}^{(2\beta\gamma)} \right]^{(p,q,s)} - Z_2 \left[ \frac{12}{d_2^2} Q_{r(1,0,0)}^{(2\beta\gamma)} \right]^{(p-1,q,s)} \\ &+ Z_3 \left[ Q_{r(0,0,0)}^{(2\beta\gamma)} - \frac{1}{R^{(2\beta\gamma)}} Q_{r(1,0,0)}^{(2\beta\gamma)} \right]^{(p,q,s)} \\ &- Z_4 \left[ Q_{r(0,0,0)}^{(2\beta\gamma)} - \frac{1}{R^{(2\beta\gamma)}} Q_{r(1,0,0)}^{(2\beta\gamma)} \right]^{(p-1,q,s)} \end{aligned} \quad (13)$$

$$\begin{aligned} \left[ Q_{r(0,0,0)}^{(1\beta\gamma)} - \frac{1}{R^{(1\beta\gamma)}} Q_{r(1,0,0)}^{(1\beta\gamma)} \right]^{(p,q,s)} &= Z_5 \left[ Q_{r(0,0,0)}^{(2\beta\gamma)} - \frac{1}{R^{(2\beta\gamma)}} Q_{r(1,0,0)}^{(2\beta\gamma)} \right]^{(p,q,s)} \\ &+ Z_6 \left[ Q_{r(0,0,0)}^{(2\beta\gamma)} - \frac{1}{R^{(2\beta\gamma)}} Q_{r(1,0,0)}^{(2\beta\gamma)} \right]^{(p-1,q,s)} \\ &- Z_7 \left[ \frac{12}{d_2^2} Q_{r(1,0,0)}^{(2\beta\gamma)} \right]^{(p,q,s)} + Z_8 \left[ \frac{12}{d_2^2} Q_{r(1,0,0)}^{(2\beta\gamma)} \right]^{(p-1,q,s)} \end{aligned} \quad (14)$$

The coefficients  $Z_i$ ,  $i = 1, \dots, 8$ , in eqns (13)-(14) are defined as follows

$$Z_1 = \frac{v_{(2\beta\gamma)}^{(p,q,s)}}{2v_{(1\beta\gamma)}^{(p,q,s)}} \quad Z_2 = \frac{v_{(2\beta\gamma)}^{(p-1,q,s)}}{2v_{(1\beta\gamma)}^{(p,q,s)}} \quad Z_3 = \frac{2Z_1}{d_2^{(p)}} \quad Z_4 = \frac{2Z_2}{d_2^{(p-1)}} \quad (15)$$

$$Z_5 = \frac{d_1^{(p)} Z_1}{d_2^{(p)}} \quad Z_6 = \frac{d_1^{(p)} Z_4}{2} \quad Z_7 = \frac{d_1^{(p)} Z_1}{2} \quad Z_8 = \frac{d_1^{(p)} Z_2}{2} \quad (16)$$

The continuity of the heat fluxes at the subcell interfaces and between individual cells in the  $\theta$ -direction, imposed in an average sense, is ensured by

$$\begin{aligned} \left[ \frac{12}{h_1^2} Q_{\theta(0,1,0)}^{(\alpha 1 \gamma)} \right]^{(p,q,s)} &= -Y_1 \left[ \frac{12}{h_2^2} Q_{\theta(0,1,0)}^{(\alpha 2 \gamma)} \right]^{(p,q,s)} - Y_2 \left[ \frac{12}{h_2^2} Q_{\theta(0,1,0)}^{(\alpha 2 \gamma)} \right]^{(p,q-1,s)} \\ &+ Y_3 \left[ Q_{\theta(0,0,0)}^{(\alpha 2 \gamma)} \right]^{(p,q,s)} - Y_4 \left[ Q_{\theta(0,0,0)}^{(\alpha 2 \gamma)} \right]^{(p,q-1,s)} \end{aligned} \quad (17)$$

$$\begin{aligned} \left[ Q_{\theta(0,0,0)}^{(\alpha 1 \gamma)} \right]^{(p,q,s)} &= Y_5 \left[ Q_{\theta(0,0,0)}^{(\alpha 2 \gamma)} \right]^{(p,q,s)} + Y_6 \left[ Q_{\theta(0,0,0)}^{(\alpha 2 \gamma)} \right]^{(p,q-1,s)} \\ &- Y_7 \left[ \frac{12}{h_2^2} Q_{\theta(0,1,0)}^{(\alpha 2 \gamma)} \right]^{(p,q,s)} + Y_8 \left[ \frac{12}{h_2^2} Q_{\theta(0,1,0)}^{(\alpha 2 \gamma)} \right]^{(p,q-1,s)} \end{aligned} \quad (18)$$

The coefficients  $Y_i$ ,  $i = 1, \dots, 8$ , in eqns (17)-(18) are defined as follows

$$Y_1 = \frac{v_{(\alpha 2 \gamma)}^{(p,q,s)}}{2v_{(\alpha 1 \gamma)}^{(p,q,s)}} \quad Y_2 = \frac{v_{(\alpha 2 \gamma)}^{(p,q-1,s)}}{2v_{(\alpha 1 \gamma)}^{(p,q,s)}} \quad Y_3 = \frac{2Y_1}{h_2^{(q)}} \quad Y_4 = \frac{2Y_2}{h_2^{(q-1)}} \quad (19)$$

$$Y_5 = \frac{h_1^{(q)} Y_1}{h_2^{(q)}} \quad Y_6 = \frac{h_1^{(q)} Y_4}{2} \quad Y_7 = \frac{h_1^{(q)} Y_1}{2} \quad Y_8 = \frac{h_1^{(q)} Y_2}{2} \quad (20)$$

Finally, the continuity of the heat fluxes at the subcell interfaces in the z-direction, imposed in an average sense, is ensured by

$$\left[ \frac{1}{l_1} Q_{z(0,0,1)}^{(\alpha \beta 1)} + \frac{1}{l_2} Q_{z(0,0,1)}^{(\alpha \beta 2)} \right]^{(p,q,s)} = 0 \quad (21)$$

The above equations (13)-(14), (17)-(18) and (21), provide us with 20 additional relations among the zeroth- and first-order heat fluxes. These 28 relations can be expressed in terms of the unknown coefficients  $T_{(lmn)}^{(\alpha \beta \gamma)}$  by making use of the expressions for heat fluxes given in terms of these coefficients in eqns (7)-(11).

An additional set of 20 equations that are necessary to determine the unknown coefficients in the temperature field expansion is subsequently generated by the thermal continuity conditions imposed on an average basis at each subcell and cell interface. Imposing the thermal continuity at each subcell interface and between individual cells in the r-direction we obtain

$$\left[ T_{(000)}^{(1\beta\gamma)} + \frac{d_1}{2} T_{(100)}^{(1\beta\gamma)} + \frac{d_1^2}{4} T_{(200)}^{(1\beta\gamma)} - T_{(000)}^{(2\beta\gamma)} + \frac{d_2}{2} T_{(100)}^{(2\beta\gamma)} - \frac{d_2^2}{4} T_{(200)}^{(2\beta\gamma)} \right]^{(p,q,s)} = 0 \quad (22)$$

$$\left[ T_{(000)}^{(2\beta\gamma)} + \frac{d_2}{2} T_{(100)}^{(2\beta\gamma)} + \frac{d_2^2}{4} T_{(200)}^{(2\beta\gamma)} \right]^{(p,q,s)} = \left[ T_{(000)}^{(1\beta\gamma)} - \frac{d_1}{2} T_{(100)}^{(1\beta\gamma)} + \frac{d_1^2}{4} T_{(200)}^{(1\beta\gamma)} \right]^{(p+1,q,s)} \quad (23)$$

In the  $\theta$ -direction we have

$$\left[ T_{(000)}^{(\alpha 1 \gamma)} + \frac{h_1}{2} T_{(010)}^{(\alpha 1 \gamma)} + \frac{h_1^2}{4} T_{(020)}^{(\alpha 1 \gamma)} - T_{(000)}^{(\alpha 2 \gamma)} + \frac{h_2}{2} T_{(010)}^{(\alpha 2 \gamma)} - \frac{h_2^2}{4} T_{(020)}^{(\alpha 2 \gamma)} \right]^{(p,q,s)} = 0 \quad (24)$$

$$[T_{(000)}^{(\alpha 2 \gamma)} + \frac{h_2}{2} T_{(010)}^{(\alpha 2 \gamma)} + \frac{h_2^2}{4} T_{(020)}^{(\alpha 2 \gamma)}]_{(p,q,s)} = [T_{(000)}^{(\alpha 1 \gamma)} - \frac{h_1}{2} T_{(010)}^{(\alpha 1 \gamma)} + \frac{h_1^2}{4} T_{(020)}^{(\alpha 1 \gamma)}]_{(p,q+1,s)} \quad (25)$$

The thermal continuity conditions in the periodic z-direction, imposed in the average sense, provide

$$[T_{(000)}^{(\alpha \beta 1)} + \frac{l_1^2}{4} T_{(002)}^{(\alpha \beta 1)} - T_{(000)}^{(\alpha \beta 2)} - \frac{l_2^2}{4} T_{(002)}^{(\alpha \beta 2)}]_{(p,q,s)} = 0 \quad (26)$$

These temperature continuity conditions, eqns (22)-(26), comprise the required additional 20 relations.

The steady-state heat equations (12) together with the heat flux, (13)-(14), (17)-(18), and thermal continuity, (22)-(26), equations form altogether 48 linear algebraic equations that govern the 48 field variables  $T_{(lmn)}^{(\alpha \beta \gamma)}$  in the eight subcells  $(\alpha \beta \gamma)$  of the interior cell  $(p, q, s)$ ,  $p = 2, \dots, N_p - 1$ ,  $q = 2, \dots, N_q - 1$ . For the boundary cells  $p = 1, N_p$  and  $q = 1, N_q$ , a different treatment must be applied. For  $p = 1$ , the flux continuity conditions (13)-(14) between a given cell and the preceding one are not applicable. They are replaced by the condition that the heat flux at the interface between subcells  $(1 \beta \gamma)$  and  $(2 \beta \gamma)$  of cell  $(1, q, s)$  is continuous as well as the applied temperature relation at the surface  $r = R_0$ . For the cell  $p = N_p$ , the previous equations are applicable except for those which express continuity between this cell and the next one, eqn (23). These equations are replaced by the boundary conditions that are applied at the surface  $r = R_1$ . In the case in which the temperature is prescribed, the boundary conditions at the bottom and top surfaces are

$$T^{(1 \beta \gamma)} |_{(1,q,s)} = T_B(\theta) \quad \bar{r}^{(1)} = -d_1^{(1)}/2 \quad (27)$$

$$T^{(2 \beta \gamma)} |_{(N_p,q,s)} = T_T(\theta) \quad \bar{r}^{(2)} = d_2^{(N_p)}/2 \quad (28)$$

where  $q = 1, \dots, N_q$ .

Similarly, continuity conditions (17)-(18) that are not applicable at  $\theta = 0$  at cell  $(p, 1, s)$  are replaced by the continuity of heat flux at the interfaces between the subcells of this cell, and by the applied loading at the surface  $\theta = 0$ . The temperature continuity conditions between a cell and the next one in the  $\theta$ -direction, eqn (25), which are not applicable at cell  $(p, N_q, s)$  are replaced by the applied loading conditions at  $\theta = \Theta$ . In the case in which the temperature is prescribed, the boundary conditions at the left and right surfaces are

$$T^{(\alpha 1 \gamma)} |_{(p,1,s)} = T_L(r) \quad \bar{y}^{(1)} = -h_1^{(1)}/2 \quad (29)$$

$$T^{(\alpha 2 \gamma)} |_{(p,N_q,s)} = T_R(r) \quad \bar{y}^{(2)} = h_2^{(N_q)}/2 \quad (30)$$

where  $p = 1, \dots, N_p$ .

The governing equations at the interior and boundary cells form a system of  $48N_pN_q$  linear algebraic equations in the unknowns  $T_{(lmn)}^{(\alpha \beta \gamma)}$ . Their solution determines the temperature distribution within the FG composite that is subjected to the specified boundary conditions. The final form of this system of equations is symbolically expressed as

$$\boldsymbol{\kappa} \mathbf{T} = \mathbf{t} \quad (31)$$

where the structural thermal conductivity matrix  $\boldsymbol{\kappa}$  contains information on the geometry and thermal conductivities of the individual subcells  $(\alpha \beta \gamma)$  in the  $N_pN_q$  cells spanning the  $r$  and  $\theta$  directions; the thermal coefficient vector  $\mathbf{T}$  contains the unknown coefficients that describe the thermal field in each subcell, i.e.,  $\mathbf{T} = [\mathbf{T}_{11}^{(111)}, \dots, \mathbf{T}_{N_p N_q}^{(222)}]$  where  $\mathbf{T}_{pq}^{(\alpha \beta \gamma)} = [T_{(000)}, T_{(100)}, T_{(010)}, T_{(200)}, T_{(020)}, T_{(002)}]_{pq}^{(\alpha \beta \gamma)}$ ; and the thermal force vector  $\mathbf{t}$  contains information on the thermal boundary conditions.

## 4. MECHANICAL ANALYSIS

**4.1. Basic Mechanical Equations.** The mechanical dynamic equations of motion in cylindrical coordinates  $(r, \theta, z)$  must be fulfilled within each  $(\alpha\beta\gamma)$  subcell of the  $(p, q, s)$ th cell. These equations are given by

$$\frac{\partial \sigma_{rr}^{(\alpha\beta\gamma)}}{\partial \bar{r}^{(\alpha)}} + \frac{\partial \sigma_{\theta r}^{(\alpha\beta\gamma)}}{\partial \bar{y}^{(\beta)}} + \frac{\partial \sigma_{zr}^{(\alpha\beta\gamma)}}{\partial \bar{z}^{(\gamma)}} + \frac{1}{R^{(\alpha\beta\gamma)} + \bar{r}^{(\alpha)}} [\sigma_{rr}^{(\alpha\beta\gamma)} - \sigma_{\theta\theta}^{(\alpha\beta\gamma)}] = \rho^{(\alpha\beta\gamma)} \frac{\partial^2 u_r^{(\alpha\beta\gamma)}}{\partial t^2} \quad (32)$$

$$\frac{\partial \sigma_{r\theta}^{(\alpha\beta\gamma)}}{\partial \bar{r}^{(\alpha)}} + \frac{\partial \sigma_{\theta\theta}^{(\alpha\beta\gamma)}}{\partial \bar{y}^{(\beta)}} + \frac{\partial \sigma_{z\theta}^{(\alpha\beta\gamma)}}{\partial \bar{z}^{(\gamma)}} + \frac{2}{R^{(\alpha\beta\gamma)} + \bar{r}^{(\alpha)}} \sigma_{r\theta}^{(\alpha\beta\gamma)} = \rho^{(\alpha\beta\gamma)} \frac{\partial^2 u_\theta^{(\alpha\beta\gamma)}}{\partial t^2} \quad (33)$$

$$\frac{\partial \sigma_{rz}^{(\alpha\beta\gamma)}}{\partial \bar{r}^{(\alpha)}} + \frac{\partial \sigma_{\theta z}^{(\alpha\beta\gamma)}}{\partial \bar{y}^{(\beta)}} + \frac{\partial \sigma_{zz}^{(\alpha\beta\gamma)}}{\partial \bar{z}^{(\gamma)}} + \frac{1}{R^{(\alpha\beta\gamma)} + \bar{r}^{(\alpha)}} \sigma_{rz}^{(\alpha\beta\gamma)} = \rho^{(\alpha\beta\gamma)} \frac{\partial^2 u_z^{(\alpha\beta\gamma)}}{\partial t^2} \quad (34)$$

where  $u_r^{(\alpha\beta\gamma)}$ ,  $u_\theta^{(\alpha\beta\gamma)}$ ,  $u_z^{(\alpha\beta\gamma)}$  are the subcell displacement components,  $\sigma_{ij}^{(\alpha\beta\gamma)}$  ( $i, j = r, \theta, z$ ) are the stress components,  $\rho^{(\alpha\beta\gamma)}$  is the mass density and  $t$  is time.

The components of the stress tensor, assuming that the material occupying the subcell  $(\alpha\beta\gamma)$  of the  $(p, q, s)$ th cell is either elastic orthotropic or inelastic isotropic, are given by

$$\sigma_{ij}^{(\alpha\beta\gamma)} = c_{ijkl}^{(\alpha\beta\gamma)} [\epsilon_{kl}^{(\alpha\beta\gamma)} - \epsilon_{kl}^{I(\alpha\beta\gamma)}] - \Gamma_{ij}^{(\alpha\beta\gamma)} T^{(\alpha\beta\gamma)} \quad (35)$$

where  $i, j, k, l = r, \theta, z$ ,  $c_{ijkl}^{(\alpha\beta\gamma)}$  are the elements of the elastic stiffness tensor,  $\epsilon_{ij}^{(\alpha\beta\gamma)}$  and  $\epsilon_{ij}^{I(\alpha\beta\gamma)}$  are the total strain and the inelastic strain components,  $T^{(\alpha\beta\gamma)}$  is the temperature, and  $\Gamma_{ij}^{(\alpha\beta\gamma)}$  are the elements of the thermal stress tensor which is the product of stiffness and the thermal expansion coefficients tensors. In this report, we consider either elastic orthotropic materials or inelastic materials which are isotropic in both elastic and inelastic domains. Hence, the above constitutive relations (35) reduce to

$$\sigma_{ij}^{(\alpha\beta\gamma)} = c_{ijkl}^{(\alpha\beta\gamma)} \epsilon_{kl}^{(\alpha\beta\gamma)} - 2\mu^{(\alpha\beta\gamma)} \epsilon_{ij}^{I(\alpha\beta\gamma)} - \sigma_{ij}^{T(\alpha\beta\gamma)} \quad (36)$$

where  $\mu^{(\alpha\beta\gamma)}$  is the elastic shear modulus of the material filling the given subcell  $(\alpha\beta\gamma)$ , and the term  $\sigma_{ij}^{T(\alpha\beta\gamma)}$  stands for the thermal contribution  $\Gamma_{ij}^{(\alpha\beta\gamma)} T^{(\alpha\beta\gamma)}$ .

**4.2. Traction continuity conditions.** The continuity of tractions between adjacent subcells within the generic cell  $(p, q, s)$  is fulfilled by requiring

$$\sigma_{ri}^{(1\beta\gamma)} \Big|_{\bar{r}^{(1)}=d_1^{(p)}/2} = \sigma_{ri}^{(2\beta\gamma)} \Big|_{\bar{r}^{(2)}=-d_2^{(p)}/2} \quad (37)$$

$$\sigma_{\theta i}^{(\alpha 1\gamma)} \Big|_{\bar{y}^{(1)}=h_1^{(q)}/2} = \sigma_{\theta i}^{(\alpha 2\gamma)} \Big|_{\bar{y}^{(2)}=-h_2^{(q)}/2} \quad (38)$$

$$\sigma_{zi}^{(\alpha\beta 1)} \Big|_{\bar{z}^{(1)}=l_1/2} = \sigma_{zi}^{(\alpha\beta 2)} \Big|_{\bar{z}^{(2)}=-l_2/2} \quad (39)$$

where  $i = r, \theta, z$ .

In addition to the above continuity conditions within the generic cell, the traction continuity at the interfaces between neighboring cells are fulfilled by satisfying

$$\sigma_{ri}^{(1\beta\gamma)} \Big|_{\bar{r}(1)=-d_1^{(p+1)}/2}^{(p+1,q,s)} = \sigma_{ri}^{(2\beta\gamma)} \Big|_{\bar{r}(2)=d_2^{(p)}/2}^{(p,q,s)} \quad (40)$$

$$\sigma_{\theta i}^{(\alpha 1\gamma)} \Big|_{\bar{y}(1)=-h_1^{(q+1)}/2}^{(p,q+1,s)} = \sigma_{\theta i}^{(\alpha 2\gamma)} \Big|_{\bar{y}(2)=h_2^{(q)}/2}^{(p,q,s)} \quad (41)$$

$$\sigma_{zi}^{(\alpha\beta 1)} \Big|_{\bar{z}(1)=-l_1/2}^{(p,q,s+1)} = \sigma_{zi}^{(\alpha\beta 2)} \Big|_{\bar{z}(2)=l_2/2}^{(p,q,s)} \quad (42)$$

**4.3. Displacement Continuity Conditions.** Similar to the traction continuity conditions described above, the following displacement continuity conditions must be satisfied at the interfaces within a generic cell  $(p, q, s)$  and its neighboring cells.

$$\mathbf{u}^{(1\beta\gamma)} \Big|_{\bar{r}(1)=d_1^{(p)}/2}^{(p,q,s)} = \mathbf{u}^{(2\beta\gamma)} \Big|_{\bar{r}(2)=-d_2^{(p)}/2}^{(p,q,s)} \quad (43)$$

$$\mathbf{u}^{(\alpha 1\gamma)} \Big|_{\bar{y}(1)=h_1^{(q)}/2}^{(p,q,s)} = \mathbf{u}^{(\alpha 2\gamma)} \Big|_{\bar{y}(2)=-h_2^{(q)}/2}^{(p,q,s)} \quad (44)$$

$$\mathbf{u}^{(\alpha\beta 1)} \Big|_{\bar{z}(1)=l_1/2}^{(p,q,s)} = \mathbf{u}^{(\alpha\beta 2)} \Big|_{\bar{z}(2)=-l_2/2}^{(p,q,s)} \quad (45)$$

where  $\mathbf{u}^{(\alpha\beta\gamma)} = (u_r^{(\alpha\beta\gamma)}, u_\theta^{(\alpha\beta\gamma)}, u_z^{(\alpha\beta\gamma)})$  denotes the displacement vector in subcell  $(\alpha\beta\gamma)$ , and

$$\mathbf{u}^{(1\beta\gamma)} \Big|_{\bar{r}(1)=-d_1^{(p+1)}/2}^{(p+1,q,s)} = \mathbf{u}^{(2\beta\gamma)} \Big|_{\bar{r}(2)=d_2^{(p)}/2}^{(p,q,s)} \quad (46)$$

$$\mathbf{u}^{(\alpha 1\gamma)} \Big|_{\bar{y}(1)=-h_1^{(q+1)}/2}^{(p,q+1,s)} = \mathbf{u}^{(\alpha 2\gamma)} \Big|_{\bar{y}(2)=h_2^{(q)}/2}^{(p,q,s)} \quad (47)$$

$$\mathbf{u}^{(\alpha\beta 1)} \Big|_{\bar{z}(1)=-l_1/2}^{(p,q,s+1)} = \mathbf{u}^{(\alpha\beta 2)} \Big|_{\bar{z}(2)=l_2/2}^{(p,q,s)} \quad (48)$$

**4.4. Boundary Conditions.** The final set of conditions that the solution for the displacement field must satisfy are the boundary conditions at the top and bottom, and left and right surfaces. For example, the tractions in cells  $(1, q, s)$  at the bottom surface  $r = R_0$  must be equal to the applied time-dependent surface loads,

$$\sigma_{rr}^{(1\beta\gamma)} \Big|_{\bar{r}(1)=-d_1^{(1)}/2}^{(1,q,s)} = f_{Br}(\theta, t) \quad (49)$$

$$\sigma_{r\theta}^{(1\beta\gamma)} \Big|_{\bar{r}(1)=-d_1^{(1)}/2}^{(1,q,s)} = f_{B\theta}(\theta, t) \quad (50)$$

where  $q = 1, \dots, N_q$ , and  $f_{Br}(\theta, t), f_{B\theta}(\theta, t)$  specify the form of these time-dependent loading functions. At the top surface  $r = R_1$

$$\sigma_{rr}^{(2\beta\gamma)} \Big|_{\bar{r}(2)=d_1^{(N_p)}/2}^{(N_p,q,s)} = f_{Tr}(\theta, t) \quad (51)$$

$$\sigma_{r\theta}^{(2\beta\gamma)} \Big|_{\bar{r}(2)=d_1^{(N_p)}/2}^{(N_p,q,s)} = f_{T\theta}(\theta, t) \quad (52)$$

where  $q = 1, \dots, N_q$ .

Similarly, the tractions in cells  $(p, 1, s)$  at the left surface  $\theta = 0$  must be equal to the applied time-dependent surface loads,

$$\sigma_{\theta r}^{(\alpha 1 \gamma)} \Big|_{\bar{y}^{(1)} = -h_1^{(1)}/2}^{(p, 1, s)} = f_{Lr}(r, t) \quad (53)$$

$$\sigma_{\theta \theta}^{(\alpha 1 \gamma)} \Big|_{\bar{y}^{(1)} = -h_1^{(1)}/2}^{(p, 1, s)} = f_{L\theta}(r, t) \quad (54)$$

where  $p = 1, \dots, N_p$ . Similar boundary conditions hold at the right surface  $\theta = \Theta$ . In the case of prescribed time-dependent displacements (say), they are:

$$u_r^{(\alpha 1 \gamma)} \Big|_{\bar{y}^{(2)} = h_2^{(N_q)}/2}^{(p, N_q, s)} = f_{Rr}(r, t) \quad (55)$$

$$u_\theta^{(\alpha 1 \gamma)} \Big|_{\bar{y}^{(2)} = h_2^{(N_q)}/2}^{(p, N_q, s)} = f_{R\theta}(r, t) \quad (56)$$

where the time-dependent loading functions are denoted by  $f_{Rr}(r, t)$  and  $f_{R\theta}(r, t)$ .

**4.5. Mechanical Field Expansion.** The time-dependent displacement components is represented in each subcell by a quadratic expansion in the local coordinates  $\bar{r}^{(\alpha)}$ ,  $\bar{y}^{(\beta)}$ ,  $\bar{z}^{(\gamma)}$  as follows:

$$\begin{aligned} u_r^{(\alpha \beta \gamma)} &= W_{1(000)}^{(\alpha \beta \gamma)} + \bar{r}^{(\alpha)} W_{1(100)}^{(\alpha \beta \gamma)} + \bar{y}^{(\beta)} W_{1(010)}^{(\alpha \beta \gamma)} \\ &+ \frac{1}{2} (3\bar{r}^{(\alpha)2} - \frac{d_\alpha^{(p)2}}{4}) W_{1(200)}^{(\alpha \beta \gamma)} + \frac{1}{2} (3\bar{y}^{(\beta)2} - \frac{h_\beta^{(q)2}}{4}) W_{1(020)}^{(\alpha \beta \gamma)} + \frac{1}{2} (3\bar{z}^{(\gamma)2} - \frac{l_\gamma^2}{4}) W_{1(002)}^{(\alpha \beta \gamma)} \end{aligned} \quad (57)$$

$$\begin{aligned} u_\theta^{(\alpha \beta \gamma)} &= W_{2(000)}^{(\alpha \beta \gamma)} + \bar{r}^{(\alpha)} W_{2(100)}^{(\alpha \beta \gamma)} + \bar{y}^{(\beta)} W_{2(010)}^{(\alpha \beta \gamma)} \\ &+ \frac{1}{2} (3\bar{r}^{(\alpha)2} - \frac{d_\alpha^{(p)2}}{4}) W_{2(200)}^{(\alpha \beta \gamma)} + \frac{1}{2} (3\bar{y}^{(\beta)2} - \frac{h_\beta^{(q)2}}{4}) W_{2(020)}^{(\alpha \beta \gamma)} + \frac{1}{2} (3\bar{z}^{(\gamma)2} - \frac{l_\gamma^2}{4}) W_{2(002)}^{(\alpha \beta \gamma)} \end{aligned} \quad (58)$$

$$u_z^{(\alpha \beta \gamma)} = W_{3(000)}^{(\alpha \beta \gamma)} + \bar{z}^{(\gamma)} W_{3(001)}^{(\alpha \beta \gamma)} \quad (59)$$

where the unknown coefficients  $W_{i(lmn)}^{(\alpha \beta \gamma)}(t)$  ( $i = 1, 2, 3$ ), which depend on time, are determined from the fulfillment of the governing equations, the interfacial traction and displacement continuity conditions, and the applied loading conditions. Note that there are 112 unknowns in eqns (57)-(59) which necessitates the establishment of 112 relations for the determination of these unknowns.

It should be noted that the  $z$ - component of the displacement field, eqn (59), does not contain linear terms in the local coordinates  $\bar{r}^{(\alpha)}$  and  $\bar{y}^{(\beta)}$ . This follows from the assumed periodicity in the axial direction and symmetry with respect to  $\bar{z}^{(\gamma)} = 0$  ( $\gamma = 1, 2$ ). Further, the presence of the constant term  $W_{3(000)}^{(\alpha \beta \gamma)}$  in eqn (59), which represents subcell center axial displacement, produces uniform composite strain  $\bar{\epsilon}_{zz}$  upon application of a partial homogenization scheme in the periodic direction described in Ref. [5]. This partial homogenization, which couples the present higher order theory and an RVE-based theory, leads to an overall behavior of a composite, functionally graded in the  $r$  and  $\theta$  directions, which can be described as a generalized plane strain in the periodic axial direction.

In the perfectly elastic case, the above quadratic displacement expansions (57)-(59) produce linear variations in strains and stresses at each point within a given subcell. In the presence of inelastic effects, however, a linear strain field generated by the above expansion does not imply the linearity of the stress field due to the path-dependent deformation. Thus the displacement field microvariables must depend implicitly on the plastic strain distributions, giving rise to a higher-order stress field than the linear strain field generated from the assumed displacement field representation. In the presence of inelastic effects, this higher-order stress field is represented by a higher order Legendre polynomial expansion in the local coordinates. Therefore, the strain field generated from the assumed displacement field, and the resulting mechanical and thermal stress fields, must be expressed in terms of Legendre polynomials as

$$\epsilon_{ij}^{(\alpha\beta\gamma)} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Lambda_{lmn} e_{ij(l,m,n)}^{(\alpha\beta\gamma)} P_l(\zeta_r^{(\alpha)}) P_m(\zeta_\theta^{(\beta)}) P_n(\zeta_z^{(\gamma)}) \quad (60)$$

$$\sigma_{ij}^{(\alpha\beta\gamma)} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Lambda_{lmn} \tau_{ij(l,m,n)}^{(\alpha\beta\gamma)} P_l(\zeta_r^{(\alpha)}) P_m(\zeta_\theta^{(\beta)}) P_n(\zeta_z^{(\gamma)}) \quad (61)$$

$$\sigma_{ij}^{T(\alpha\beta\gamma)} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Lambda_{lmn} \tau_{ij(l,m,n)}^{T(\alpha\beta\gamma)} P_l(\zeta_r^{(\alpha)}) P_m(\zeta_\theta^{(\beta)}) P_n(\zeta_z^{(\gamma)}) \quad (62)$$

where  $i, j = r, \theta, z$ ,  $\Lambda_{lmn} = \sqrt{(2l+1)(2m+1)(2n+1)}$ , and the nondimensionalized variables  $\zeta_i$ 's defined in the interval  $-1 \leq \zeta_i \leq 1$ , are given in terms of the local subcell coordinates as  $\zeta_r^{(\alpha)} = \bar{r}^{(\alpha)}/(d_\alpha/2)$ ,  $\zeta_\theta^{(\beta)} = \bar{y}^{(\beta)}/(h_\beta/2)$ ,  $\zeta_z^{(\gamma)} = \bar{z}^{(\gamma)}/(l_\gamma/2)$ .

For the given displacement field representation the upper limits on the summation in the strain expansion (60) becomes 1; while for a quadratic temperature distribution, the upper limits in the thermal stress expansion (62) become 2. Alternatively, the upper limits on the summations in the stress expansion (61) are chosen so that an accurate representation of the stress fields is obtained within each subcell, which depends on the amount of plastic flow. The coefficients  $e_{ij(l,m,n)}^{(\alpha\beta\gamma)}$ ,  $\tau_{ij(l,m,n)}^{(\alpha\beta\gamma)}$ ,  $\tau_{ij(l,m,n)}^{T(\alpha\beta\gamma)}$  in the above expansions are determined as described below.

The strain coefficients  $e_{ij(l,m,n)}^{(\alpha\beta\gamma)}$  are explicitly determined in terms of the displacement field microvariables using orthogonal properties of Legendre polynomials. For example  $e_{rr(0,0,0)}^{(\alpha\beta\gamma)} = W_{1(100)}^{(\alpha\beta\gamma)}$ . The complete set of nonzero strain coefficients is given in the Appendix.

Similarly, the thermal stress coefficients  $\tau_{ij(l,m,n)}^{T(\alpha\beta\gamma)}$  can be expressed in terms of the temperature field microvariables  $T_{(lmn)}^{(\alpha\beta\gamma)}$ . For example  $\tau_{rr(0,0,0)}^{T(\alpha\beta\gamma)} = \Gamma_{rr}^{(\alpha\beta\gamma)} T_{(000)}^{(\alpha\beta\gamma)}$ . The complete set of nonzero thermal stress coefficients is also given in the Appendix.

The stress coefficients  $\tau_{ij(l,m,n)}^{(\alpha\beta\gamma)}$  are expressed in terms of strain coefficients, the thermal stress coefficients, and the unknown inelastic strain distributions by first substituting the Legendre polynomial representations for  $\epsilon_{ij}^{(\alpha\beta\gamma)}$ ,  $\sigma_{ij}^{(\alpha\beta\gamma)}$ , and  $\sigma_{ij}^{T(\alpha\beta\gamma)}$  into the constitutive equations (36) and then utilizing the orthogonality of Legendre polynomials. This yields

$$\tau_{ij(l,m,n)}^{(\alpha\beta\gamma)} = c_{ijk\alpha}^{(\alpha\beta\gamma)} e_{ko(l,m,n)}^{(\alpha\beta\gamma)} - \tau_{ij(l,m,n)}^{T(\alpha\beta\gamma)} - R_{ij(l,m,n)}^{(\alpha\beta\gamma)} \quad (63)$$

The  $R_{ij(l,m,n)}^{(\alpha\beta\gamma)}$  terms depend on the inelastic strain distributions calculated in the following manner:

$$R_{ij(l,m,n)}^{(\alpha\beta\gamma)} = \mu_{((\alpha\beta\gamma))} \frac{\Lambda_{lmn}}{4} \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \epsilon_{ij}^{I(\alpha\beta\gamma)} P_l(\zeta_r^{(\alpha)}) P_m(\zeta_\theta^{(\beta)}) P_n(\zeta_z^{(\gamma)}) d\zeta_r^{(\alpha)} d\zeta_\theta^{(\beta)} d\zeta_z^{(\gamma)} \quad (64)$$

**4.6. Stress Moments.** Let us define the following stress quantities in subcell  $(\alpha\beta\gamma)$  of cell  $(p, q, s)$

$$[S_{ij(l,m,n)}^{(\alpha\beta\gamma)}]^{(p,q,s)} = \frac{1}{v_{(\alpha\beta\gamma)}^{(p,q,s)}} \int_{-d_\alpha^{(p)}/2}^{d_\alpha^{(p)}/2} \int_{-h_\beta^{(q)}/2}^{h_\beta^{(q)}/2} \int_{-l_\gamma/2}^{l_\gamma/2} (\bar{r}^{(\alpha)})^l (\bar{y}^{(\beta)})^m (\bar{z}^{(\gamma)})^n \sigma_{ij}^{(\alpha\beta\gamma)} d\bar{r}^{(\alpha)} d\bar{y}^{(\beta)} d\bar{z}^{(\gamma)} \quad (65)$$

where  $i, j = r, \theta, z$ . Note that, in particular, the zero-order quantities  $S_{ij(0,0,0)}^{(\alpha\beta\gamma)}$  represent the average stresses in the subcell.

Explicit evaluation of eqn (65) yields the following expressions for the normal stress quantities (superscripts  $(p, q, s)$  have been omitted)

$$\begin{aligned} S_{rr(0,0,0)}^{(\alpha\beta\gamma)} &= c_{11}^{(\alpha\beta\gamma)} W_{1(100)}^{(\alpha\beta\gamma)} + c_{12}^{(\alpha\beta\gamma)} [W_{2(010)}^{(\alpha\beta\gamma)} + \frac{1}{R(\alpha\beta\gamma)} W_{1(000)}^{(\alpha\beta\gamma)}] + c_{13}^{(\alpha\beta\gamma)} W_{3(001)}^{(\alpha\beta\gamma)} \\ &- \Gamma_{rr}^{(\alpha\beta\gamma)} T_{(000)}^{(\alpha\beta\gamma)} - R_{rr(0,0,0)}^{(\alpha\beta\gamma)} \end{aligned} \quad (66)$$

with similar expressions for  $S_{\theta\theta(0,0,0)}^{(\alpha\beta\gamma)}$  and  $S_{zz(0,0,0)}^{(\alpha\beta\gamma)}$ .

$$\begin{aligned} S_{rr(1,0,0)}^{(\alpha\beta\gamma)} &= \frac{d_\alpha^2}{4} c_{11}^{(\alpha\beta\gamma)} W_{1(200)}^{(\alpha\beta\gamma)} + \frac{d_\alpha^2}{12R(\alpha\beta\gamma)} c_{12}^{(\alpha\beta\gamma)} W_{1(100)}^{(\alpha\beta\gamma)} \\ &- \frac{d_\alpha^2}{12} \Gamma_{rr}^{(\alpha\beta\gamma)} T_{(100)}^{(\alpha\beta\gamma)} - \frac{d_\alpha}{2\sqrt{3}} R_{rr(1,0,0)}^{(\alpha\beta\gamma)} \end{aligned} \quad (67)$$

with similar expressions for  $S_{\theta\theta(1,0,0)}^{(\alpha\beta\gamma)}$  and  $S_{zz(1,0,0)}^{(\alpha\beta\gamma)}$ .

$$\begin{aligned} S_{rr(0,1,0)}^{(\alpha\beta\gamma)} &= \frac{h_\beta^2}{4} c_{12}^{(\alpha\beta\gamma)} W_{2(020)}^{(\alpha\beta\gamma)} + \frac{h_\beta^2}{12R(\alpha\beta\gamma)} c_{12}^{(\alpha\beta\gamma)} W_{1(010)}^{(\alpha\beta\gamma)} \\ &- \frac{h_\beta^2}{12} \Gamma_{rr}^{(\alpha\beta\gamma)} T_{(010)}^{(\alpha\beta\gamma)} - \frac{h_\beta}{2\sqrt{3}} R_{rr(0,1,0)}^{(\alpha\beta\gamma)} \end{aligned} \quad (68)$$

with similar expressions for  $S_{\theta\theta(0,1,0)}^{(\alpha\beta\gamma)}$  and  $S_{zz(0,1,0)}^{(\alpha\beta\gamma)}$ .

$$S_{rr(2,0,0)}^{(\alpha\beta\gamma)} = \frac{d_\alpha^2}{12} [S_{rr(0,0,0)}^{(\alpha\beta\gamma)} + \frac{d_\alpha^2}{10} (\frac{c_{12}^{(\alpha\beta\gamma)}}{R(\alpha\beta\gamma)} W_{1(200)}^{(\alpha\beta\gamma)} - \Gamma_{rr}^{(\alpha\beta\gamma)} T_{(200)}^{(\alpha\beta\gamma)}) - \frac{2}{\sqrt{5}} R_{rr(2,0,0)}^{(\alpha\beta\gamma)}] \quad (69)$$

$$S_{rr(0,2,0)}^{(\alpha\beta\gamma)} = \frac{h_\beta^2}{12} [S_{rr(0,0,0)}^{(\alpha\beta\gamma)} + \frac{h_\beta^2}{10} (\frac{c_{12}^{(\alpha\beta\gamma)}}{R(\alpha\beta\gamma)} W_{1(020)}^{(\alpha\beta\gamma)} - \Gamma_{rr}^{(\alpha\beta\gamma)} T_{(020)}^{(\alpha\beta\gamma)}) - \frac{2}{\sqrt{5}} R_{rr(0,2,0)}^{(\alpha\beta\gamma)}] \quad (70)$$

$$S_{rr(0,0,2)}^{(\alpha\beta\gamma)} = \frac{l_\gamma^2}{12} [S_{rr(0,0,0)}^{(\alpha\beta\gamma)} + \frac{l_\gamma^2}{10} (\frac{c_{12}^{(\alpha\beta\gamma)}}{R(\alpha\beta\gamma)} W_{1(002)}^{(\alpha\beta\gamma)} - \Gamma_{rr}^{(\alpha\beta\gamma)} T_{(002)}^{(\alpha\beta\gamma)}) - \frac{2}{\sqrt{5}} R_{rr(0,0,2)}^{(\alpha\beta\gamma)}] \quad (71)$$

with similar expressions for the other components.

Similarly, the explicit expressions for the shear stress quantities are

$$S_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} = c_{44}^{(\alpha\beta\gamma)} [W_{2(100)}^{(\alpha\beta\gamma)} + W_{1(010)}^{(\alpha\beta\gamma)} - \frac{1}{R^{(\alpha\beta\gamma)}} W_{2(000)}^{(\alpha\beta\gamma)}] - R_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} \quad (72)$$

$$S_{r\theta(1,0,0)}^{(\alpha\beta\gamma)} = \frac{d_\alpha^2}{4} c_{44}^{(\alpha\beta\gamma)} [W_{2(200)}^{(\alpha\beta\gamma)} - \frac{1}{3R^{(\alpha\beta\gamma)}} W_{2(100)}^{(\alpha\beta\gamma)}] - \frac{d_\alpha}{2\sqrt{3}} R_{r\theta(1,0,0)}^{(\alpha\beta\gamma)} \quad (73)$$

$$S_{r\theta(0,1,0)}^{(\alpha\beta\gamma)} = \frac{h_\beta^2}{4} c_{44}^{(\alpha\beta\gamma)} [W_{1(020)}^{(\alpha\beta\gamma)} - \frac{1}{3R^{(\alpha\beta\gamma)}} W_{2(010)}^{(\alpha\beta\gamma)}] - \frac{h_\beta}{2\sqrt{3}} R_{r\theta(0,1,0)}^{(\alpha\beta\gamma)} \quad (74)$$

$$S_{r\theta(2,0,0)}^{(\alpha\beta\gamma)} = \frac{d_\alpha^2}{12} [S_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} - \frac{d_\alpha^2}{10R^{(\alpha\beta\gamma)}} c_{44}^{(\alpha\beta\gamma)} W_{2(200)}^{(\alpha\beta\gamma)} - \frac{2}{\sqrt{5}} R_{r\theta(2,0,0)}^{(\alpha\beta\gamma)}] \quad (75)$$

$$S_{r\theta(0,2,0)}^{(\alpha\beta\gamma)} = \frac{h_\beta^2}{12} [S_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} - \frac{h_\beta^2}{10R^{(\alpha\beta\gamma)}} c_{44}^{(\alpha\beta\gamma)} W_{2(020)}^{(\alpha\beta\gamma)} - \frac{2}{\sqrt{5}} R_{r\theta(0,2,0)}^{(\alpha\beta\gamma)}] \quad (76)$$

$$S_{r\theta(0,0,2)}^{(\alpha\beta\gamma)} = \frac{l_\gamma^2}{12} [S_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} - \frac{l_\gamma^2}{10R^{(\alpha\beta\gamma)}} c_{44}^{(\alpha\beta\gamma)} W_{2(002)}^{(\alpha\beta\gamma)} - \frac{2}{\sqrt{5}} R_{r\theta(0,0,2)}^{(\alpha\beta\gamma)}] \quad (77)$$

$$S_{rz(0,0,1)}^{(\alpha\beta\gamma)} = \frac{l_\gamma^2}{4} c_{44}^{(\alpha\beta\gamma)} W_{1(002)}^{(\alpha\beta\gamma)} - \frac{l_\gamma}{2\sqrt{3}} R_{rz(0,0,1)}^{(\alpha\beta\gamma)} \quad (78)$$

$$S_{\theta z(0,0,1)}^{(\alpha\beta\gamma)} = \frac{l_\gamma^2}{4} c_{44}^{(\alpha\beta\gamma)} W_{2(002)}^{(\alpha\beta\gamma)} - \frac{l_\gamma}{2\sqrt{3}} R_{\theta z(0,0,1)}^{(\alpha\beta\gamma)} \quad (79)$$

**4.7. Zero-Moments of the Equations of Motion.** By integrating the three equations of motion (32)-(34) over the subcell  $(\alpha\beta\gamma)$  we obtain, in conjunction with the above displacement expansions (57)-(59),

$$I_{rr(0,0,0)}^{(\alpha\beta\gamma)} + J_{\theta r(0,0,0)}^{(\alpha\beta\gamma)} + K_{zr(0,0,0)}^{(\alpha\beta\gamma)} + \frac{1}{R^{(\alpha\beta\gamma)}} [S_{rr(0,0,0)}^{(\alpha\beta\gamma)} - S_{\theta\theta(0,0,0)}^{(\alpha\beta\gamma)}] = \rho^{(\alpha\beta\gamma)} \ddot{W}_{1(000)}^{(\alpha\beta\gamma)} \quad (80)$$

$$I_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} + J_{\theta\theta(0,0,0)}^{(\alpha\beta\gamma)} + K_{z\theta(0,0,0)}^{(\alpha\beta\gamma)} + \frac{2}{R^{(\alpha\beta\gamma)}} S_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} = \rho^{(\alpha\beta\gamma)} \ddot{W}_{2(000)}^{(\alpha\beta\gamma)} \quad (81)$$

$$I_{rz(0,0,0)}^{(\alpha\beta\gamma)} + J_{\theta z(0,0,0)}^{(\alpha\beta\gamma)} + K_{zz(0,0,0)}^{(\alpha\beta\gamma)} + \frac{1}{R^{(\alpha\beta\gamma)}} S_{rz(0,0,0)}^{(\alpha\beta\gamma)} = 0 \quad (82)$$

where the following interfacial traction integrals have been defined:

$$I_{rj(n,0,0)}^{(\alpha\beta\gamma)} = \frac{1}{v_{(\alpha\beta\gamma)}} \left(\frac{d_\alpha}{2}\right)^n \int_{-h_\beta/2}^{h_\beta/2} \int_{-l_\gamma/2}^{l_\gamma/2} [\sigma_{rj}^{(\alpha\beta\gamma)} \left(\frac{d_\alpha}{2}\right) + (-1)^{n+1} \sigma_{rj}^{(\alpha\beta\gamma)} \left(-\frac{d_\alpha}{2}\right)] d\bar{y}^{(\beta)} d\bar{z}^{(\gamma)} \quad (83)$$

$$J_{\theta j(0,n,0)}^{(\alpha\beta\gamma)} = \frac{1}{v_{(\alpha\beta\gamma)}} \left(\frac{h_\beta}{2}\right)^n \int_{-d_\alpha/2}^{d_\alpha/2} \int_{-l_\gamma/2}^{l_\gamma/2} [\sigma_{\theta j}^{(\alpha\beta\gamma)} \left(\frac{h_\beta}{2}\right) + (-1)^{n+1} \sigma_{\theta j}^{(\alpha\beta\gamma)} \left(-\frac{h_\beta}{2}\right)] d\bar{r}^{(\alpha)} d\bar{z}^{(\gamma)} \quad (84)$$

$$K_{zj(0,0,n)}^{(\alpha\beta\gamma)} = \frac{1}{v_{(\alpha\beta\gamma)}} \left(\frac{l_\gamma}{2}\right)^n \int_{-d_\alpha/2}^{d_\alpha/2} \int_{-h_\beta/2}^{h_\beta/2} [\sigma_{zj}^{(\alpha\beta\gamma)} \left(\frac{l_\gamma}{2}\right) + (-1)^{n+1} \sigma_{zj}^{(\alpha\beta\gamma)} \left(-\frac{l_\gamma}{2}\right)] d\bar{r}^{(\alpha)} d\bar{y}^{(\beta)} \quad (85)$$

where  $j = r, \theta, z$ ;  $n = 0$  or  $1$ , and  $\sigma_{rj}^{(\alpha\beta\gamma)}(\pm d_\alpha/2)$ ,  $\sigma_{\theta j}^{(\alpha\beta\gamma)}(\pm h_\beta/2)$ ,  $\sigma_{zj}^{(\alpha\beta\gamma)}(\pm l_\gamma/2)$ , stand for the interfacial stresses at  $\pm d_\alpha/2$ ,  $\pm h_\beta/2$ ,  $\pm l_\gamma/2$ , respectively.

**4.8. First-Moments of the Equations of Motion.** By multiplying the three equations of motion (32)-(34) by  $\bar{r}^{(\alpha)}$ , and integrating over the subcell volume by parts, we obtain:

$$I_{rr(1,0,0)}^{(\alpha\beta\gamma)} - S_{rr(0,0,0)}^{(\alpha\beta\gamma)} + \frac{1}{R(\alpha\beta\gamma)} [S_{rr(1,0,0)}^{(\alpha\beta\gamma)} - S_{\theta\theta(1,0,0)}^{(\alpha\beta\gamma)}] = \rho^{(\alpha\beta\gamma)} \frac{d_\alpha^2}{12} \ddot{W}_{1(100)}^{(\alpha\beta\gamma)} \quad (86)$$

$$I_{r\theta(1,0,0)}^{(\alpha\beta\gamma)} - S_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} + \frac{2}{R(\alpha\beta\gamma)} S_{r\theta(1,0,0)}^{(\alpha\beta\gamma)} = \rho^{(\alpha\beta\gamma)} \frac{d_\alpha^2}{12} \ddot{W}_{2(100)}^{(\alpha\beta\gamma)} \quad (87)$$

$$I_{rz(1,0,0)}^{(\alpha\beta\gamma)} - S_{rz(0,0,0)}^{(\alpha\beta\gamma)} + \frac{1}{R(\alpha\beta\gamma)} S_{rz(1,0,0)}^{(\alpha\beta\gamma)} = 0 \quad (88)$$

By multiplying the three equations of motion (32)-(34) by  $\bar{y}^{(\beta)}$ , and integrating over the subcell volume by parts, we obtain:

$$J_{\theta r(0,1,0)}^{(\alpha\beta\gamma)} - S_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} + \frac{1}{R(\alpha\beta\gamma)} [S_{rr(0,1,0)}^{(\alpha\beta\gamma)} - S_{\theta\theta(0,1,0)}^{(\alpha\beta\gamma)}] = \rho^{(\alpha\beta\gamma)} \frac{h_\beta^2}{12} \ddot{W}_{1(010)}^{(\alpha\beta\gamma)} \quad (89)$$

$$J_{\theta\theta(0,1,0)}^{(\alpha\beta\gamma)} - S_{\theta\theta(0,0,0)}^{(\alpha\beta\gamma)} + \frac{2}{R(\alpha\beta\gamma)} S_{r\theta(0,1,0)}^{(\alpha\beta\gamma)} = \rho^{(\alpha\beta\gamma)} \frac{h_\beta^2}{12} \ddot{W}_{2(010)}^{(\alpha\beta\gamma)} \quad (90)$$

$$J_{\theta z(0,1,0)}^{(\alpha\beta\gamma)} - S_{\theta z(0,0,0)}^{(\alpha\beta\gamma)} + \frac{1}{R(\alpha\beta\gamma)} S_{rz(0,1,0)}^{(\alpha\beta\gamma)} = 0 \quad (91)$$

By multiplying the three equations of motion (32)-(34) by  $\bar{z}^{(\gamma)}$ , and integrating over the subcell volume by parts, we obtain:

$$K_{zr(0,0,1)}^{(\alpha\beta\gamma)} - S_{rz(0,0,0)}^{(\alpha\beta\gamma)} + \frac{1}{R(\alpha\beta\gamma)} [S_{rr(0,0,1)}^{(\alpha\beta\gamma)} - S_{\theta\theta(0,0,1)}^{(\alpha\beta\gamma)}] = 0 \quad (92)$$

$$K_{z\theta(0,0,1)}^{(\alpha\beta\gamma)} - S_{\theta z(0,0,0)}^{(\alpha\beta\gamma)} + \frac{2}{R(\alpha\beta\gamma)} S_{r\theta(0,0,1)}^{(\alpha\beta\gamma)} = 0 \quad (93)$$

$$K_{zz(0,0,1)}^{(\alpha\beta\gamma)} - S_{zz(0,0,0)}^{(\alpha\beta\gamma)} + \frac{1}{R(\alpha\beta\gamma)} S_{rz(0,0,1)}^{(\alpha\beta\gamma)} = \rho^{(\alpha\beta\gamma)} \frac{l_\gamma^2}{12} \ddot{W}_{3(001)}^{(\alpha\beta\gamma)} \quad (94)$$

**4.9. Second-Moments of the Equations of Motion.** By multiplying the three equations of motion (32)-(34) by  $\bar{r}^{(\alpha)2}$ , and integrating over the subcell volume by parts, we obtain:

$$\begin{aligned} \frac{d_\alpha^2}{4} I_{rr(0,0,0)}^{(\alpha\beta\gamma)} + \frac{d_\alpha^2}{12} [J_{\theta r(0,0,0)}^{(\alpha\beta\gamma)} + K_{zr(0,0,0)}^{(\alpha\beta\gamma)}] - 2S_{rr(1,0,0)}^{(\alpha\beta\gamma)} \\ + \frac{1}{R(\alpha\beta\gamma)} [S_{rr(2,0,0)}^{(\alpha\beta\gamma)} - S_{\theta\theta(2,0,0)}^{(\alpha\beta\gamma)}] = \rho^{(\alpha\beta\gamma)} \frac{d_\alpha^2}{12} [\ddot{W}_{1(000)}^{(\alpha\beta\gamma)} + \frac{d_\alpha^2}{10} \ddot{W}_{1(200)}^{(\alpha\beta\gamma)}] \end{aligned} \quad (95)$$

$$\begin{aligned} \frac{d_\alpha^2}{4} I_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} + \frac{d_\alpha^2}{12} [J_{\theta\theta(0,0,0)}^{(\alpha\beta\gamma)} + K_{z\theta(0,0,0)}^{(\alpha\beta\gamma)}] - 2S_{r\theta(1,0,0)}^{(\alpha\beta\gamma)} \\ + \frac{2}{R(\alpha\beta\gamma)} S_{r\theta(2,0,0)}^{(\alpha\beta\gamma)} = \rho^{(\alpha\beta\gamma)} \frac{d_\alpha^2}{12} [\ddot{W}_{2(000)}^{(\alpha\beta\gamma)} + \frac{d_\alpha^2}{10} \ddot{W}_{2(200)}^{(\alpha\beta\gamma)}] \end{aligned} \quad (96)$$

$$\frac{d_\alpha^2}{4} I_{rz(0,0,0)}^{(\alpha\beta\gamma)} + \frac{d_\alpha^2}{12} [J_{\theta z(0,0,0)}^{(\alpha\beta\gamma)} + K_{zz(0,0,0)}^{(\alpha\beta\gamma)}] - 2S_{rz(1,0,0)}^{(\alpha\beta\gamma)} + \frac{1}{R(\alpha\beta\gamma)} S_{rz(2,0,0)}^{(\alpha\beta\gamma)} = 0 \quad (97)$$

By multiplying the three equations of motion (32)-(34) by  $\bar{y}^{(\beta)2}$ , and integrating over the subcell volume by parts, we obtain:

$$\begin{aligned} \frac{h_\beta^2}{4} J_{\theta r(0,0,0)}^{(\alpha\beta\gamma)} + \frac{h_\beta^2}{12} [I_{rr(0,0,0)}^{(\alpha\beta\gamma)} + K_{zr(0,0,0)}^{(\alpha\beta\gamma)}] - 2S_{r\theta(0,1,0)}^{(\alpha\beta\gamma)} \\ + \frac{1}{R^{(\alpha\beta\gamma)}} [S_{rr(0,2,0)}^{(\alpha\beta\gamma)} - S_{\theta\theta(0,2,0)}^{(\alpha\beta\gamma)}] = \rho^{(\alpha\beta\gamma)} \frac{h_\beta^2}{12} [\ddot{W}_{1(000)}^{(\alpha\beta\gamma)} + \frac{h_\beta^2}{10} \ddot{W}_{1(020)}^{(\alpha\beta\gamma)}] \end{aligned} \quad (98)$$

$$\begin{aligned} \frac{h_\beta^2}{4} J_{\theta\theta(0,0,0)}^{(\alpha\beta\gamma)} + \frac{h_\beta^2}{12} [I_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} + K_{z\theta(0,0,0)}^{(\alpha\beta\gamma)}] - 2S_{\theta\theta(0,1,0)}^{(\alpha\beta\gamma)} \\ + \frac{2}{R^{(\alpha\beta\gamma)}} S_{r\theta(0,2,0)}^{(\alpha\beta\gamma)} = \rho^{(\alpha\beta\gamma)} \frac{h_\beta^2}{12} [\ddot{W}_{2(000)}^{(\alpha\beta\gamma)} + \frac{h_\beta^2}{10} \ddot{W}_{2(020)}^{(\alpha\beta\gamma)}] \end{aligned} \quad (99)$$

$$\frac{h_\beta^2}{4} J_{\theta z(0,0,0)}^{(\alpha\beta\gamma)} + \frac{h_\beta^2}{12} [I_{rz(0,0,0)}^{(\alpha\beta\gamma)} + K_{zz(0,0,0)}^{(\alpha\beta\gamma)}] - 2S_{\theta z(0,1,0)}^{(\alpha\beta\gamma)} + \frac{1}{R^{(\alpha\beta\gamma)}} S_{rz(0,2,0)}^{(\alpha\beta\gamma)} = 0 \quad (100)$$

By multiplying the three equations of motion (32)-(34) by  $\bar{z}^{(\gamma)2}$ , and integrating over the subcell volume by parts, we obtain:

$$\begin{aligned} \frac{l_\gamma^2}{4} K_{zr(0,0,0)}^{(\alpha\beta\gamma)} + \frac{l_\gamma^2}{12} [I_{rr(0,0,0)}^{(\alpha\beta\gamma)} + J_{\theta r(0,0,0)}^{(\alpha\beta\gamma)}] - 2S_{rz(0,0,1)}^{(\alpha\beta\gamma)} \\ + \frac{1}{R^{(\alpha\beta\gamma)}} [S_{rr(0,0,2)}^{(\alpha\beta\gamma)} - S_{\theta\theta(0,0,2)}^{(\alpha\beta\gamma)}] = \rho^{(\alpha\beta\gamma)} \frac{l_\gamma^2}{12} [\ddot{W}_{1(000)}^{(\alpha\beta\gamma)} + \frac{l_\gamma^2}{10} \ddot{W}_{1(002)}^{(\alpha\beta\gamma)}] \end{aligned} \quad (101)$$

$$\begin{aligned} \frac{l_\gamma^2}{4} K_{z\theta(0,0,0)}^{(\alpha\beta\gamma)} + \frac{l_\gamma^2}{12} [I_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} + J_{\theta\theta(0,0,0)}^{(\alpha\beta\gamma)}] - 2S_{\theta z(0,0,1)}^{(\alpha\beta\gamma)} \\ + \frac{2}{R^{(\alpha\beta\gamma)}} S_{r\theta(0,0,2)}^{(\alpha\beta\gamma)} = \rho^{(\alpha\beta\gamma)} \frac{l_\gamma^2}{12} [\ddot{W}_{2(000)}^{(\alpha\beta\gamma)} + \frac{l_\gamma^2}{10} \ddot{W}_{2(002)}^{(\alpha\beta\gamma)}] \end{aligned} \quad (102)$$

$$\frac{l_\gamma^2}{4} K_{zz(0,0,0)}^{(\alpha\beta\gamma)} + \frac{l_\gamma^2}{12} [I_{rz(0,0,0)}^{(\alpha\beta\gamma)} + J_{\theta z(0,0,0)}^{(\alpha\beta\gamma)}] - 2S_{zz(0,0,1)}^{(\alpha\beta\gamma)} + \frac{1}{R^{(\alpha\beta\gamma)}} S_{rz(0,0,2)}^{(\alpha\beta\gamma)} = 0 \quad (103)$$

**4.10. Interfacial Traction Integrals.** From the above 21 relations (80)-(82) and (86)-(103) it is possible to establish expressions for the interfacial traction integrals  $I_{rj(0,0,0)}^{(\alpha\beta\gamma)}$ ,  $J_{\theta j(0,0,0)}^{(\alpha\beta\gamma)}$ ,  $K_{zj(0,0,0)}^{(\alpha\beta\gamma)}$ ,  $j = r, \theta, z$ , as follows.

Substitution of eqn (80) into eqn (95) yields the following expression for  $I_{rr(0,0,0)}^{(\alpha\beta\gamma)}$ :

$$\begin{aligned} I_{rr(0,0,0)}^{(\alpha\beta\gamma)} &= \frac{12}{d_\alpha^2} S_{rr(1,0,0)}^{(\alpha\beta\gamma)} + \frac{1}{2R^{(\alpha\beta\gamma)}} [S_{rr(0,0,0)}^{(\alpha\beta\gamma)} - S_{\theta\theta(0,0,0)}^{(\alpha\beta\gamma)}] \\ &\quad - \frac{6}{d_\alpha^2 R^{(\alpha\beta\gamma)}} [S_{rr(2,0,0)}^{(\alpha\beta\gamma)} - S_{\theta\theta(2,0,0)}^{(\alpha\beta\gamma)}] + \rho^{(\alpha\beta\gamma)} \frac{d_\alpha^2}{20} \ddot{W}_{1(200)}^{(\alpha\beta\gamma)} \end{aligned} \quad (104)$$

Similarly, substitution of eqn (81) into eqn (96) yields

$$\begin{aligned}
I_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} &= \frac{12}{d_\alpha^2} S_{r\theta(1,0,0)}^{(\alpha\beta\gamma)} + \frac{1}{R^{(\alpha\beta\gamma)}} S_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} \\
&- \frac{12}{d_\alpha^2 R^{(\alpha\beta\gamma)}} S_{r\theta(2,0,0)}^{(\alpha\beta\gamma)} + \rho^{(\alpha\beta\gamma)} \frac{d_\alpha^2}{20} \ddot{W}_{2(200)}^{(\alpha\beta\gamma)}
\end{aligned} \tag{105}$$

Substitution of eqn (82) into eqn (97) yields

$$I_{rz(0,0,0)}^{(\alpha\beta\gamma)} = 0 \tag{106}$$

Substitution of eqn (80) into eqn (98) yields the following expression for  $J_{\theta r(0,0,0)}^{(\alpha\beta\gamma)}$ :

$$\begin{aligned}
J_{\theta r(0,0,0)}^{(\alpha\beta\gamma)} &= \frac{12}{h_\beta^2} S_{r\theta(0,1,0)}^{(\alpha\beta\gamma)} + \frac{1}{2R^{(\alpha\beta\gamma)}} [S_{rr(0,0,0)}^{(\alpha\beta\gamma)} - S_{\theta\theta(0,0,0)}^{(\alpha\beta\gamma)}] \\
&- \frac{6}{h_\beta^2 R^{(\alpha\beta\gamma)}} [S_{rr(0,2,0)}^{(\alpha\beta\gamma)} - S_{\theta\theta(0,2,0)}^{(\alpha\beta\gamma)}] + \rho^{(\alpha\beta\gamma)} \frac{h_\beta^2}{20} \ddot{W}_{1(020)}^{(\alpha\beta\gamma)}
\end{aligned} \tag{107}$$

Similarly, substitution of eqn (81) into eqn (99) yields

$$\begin{aligned}
J_{\theta\theta(0,0,0)}^{(\alpha\beta\gamma)} &= \frac{12}{h_\beta^2} S_{\theta\theta(0,1,0)}^{(\alpha\beta\gamma)} + \frac{1}{R^{(\alpha\beta\gamma)}} S_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} \\
&- \frac{12}{h_\beta^2 R^{(\alpha\beta\gamma)}} S_{r\theta(0,2,0)}^{(\alpha\beta\gamma)} + \rho^{(\alpha\beta\gamma)} \frac{h_\beta^2}{20} \ddot{W}_{2(020)}^{(\alpha\beta\gamma)}
\end{aligned} \tag{108}$$

Substitution of eqn (82) into eqn (100) yields

$$J_{\theta z(0,0,0)}^{(\alpha\beta\gamma)} = 0 \tag{109}$$

Substitution of eqn (80) into eqn (101) yields the following expression for  $K_{zr(0,0,0)}^{(\alpha\beta\gamma)}$ :

$$\begin{aligned}
K_{zr(0,0,0)}^{(\alpha\beta\gamma)} &= \frac{12}{l_\gamma^2} S_{rz(0,0,1)}^{(\alpha\beta\gamma)} + \frac{1}{2R^{(\alpha\beta\gamma)}} [S_{rr(0,0,0)}^{(\alpha\beta\gamma)} - S_{\theta\theta(0,0,0)}^{(\alpha\beta\gamma)}] \\
&- \frac{6}{l_\gamma^2 R^{(\alpha\beta\gamma)}} [S_{rr(0,0,2)}^{(\alpha\beta\gamma)} - S_{\theta\theta(0,0,2)}^{(\alpha\beta\gamma)}] + \rho^{(\alpha\beta\gamma)} \frac{l_\gamma^2}{20} \ddot{W}_{1(002)}^{(\alpha\beta\gamma)}
\end{aligned} \tag{110}$$

Similarly, substitution of eqn (81) into eqn (102) yields

$$\begin{aligned}
K_{z\theta(0,0,0)}^{(\alpha\beta\gamma)} &= \frac{12}{l_\gamma^2} S_{\theta z(0,0,1)}^{(\alpha\beta\gamma)} + \frac{1}{R^{(\alpha\beta\gamma)}} S_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} \\
&- \frac{12}{l_\gamma^2 R^{(\alpha\beta\gamma)}} S_{r\theta(0,0,2)}^{(\alpha\beta\gamma)} + \rho^{(\alpha\beta\gamma)} \frac{l_\gamma^2}{20} \ddot{W}_{2(002)}^{(\alpha\beta\gamma)}
\end{aligned} \tag{111}$$

Substitution of eqn (82) into eqn (103) yields

$$K_{zz(0,0,0)}^{(\alpha\beta\gamma)} = 0 \tag{112}$$

It should be noted that the remaining nine expressions for the interfacial traction integrals  $I_{rj}^{(\alpha\beta\gamma)}$ ,  $J_{\theta j}^{(\alpha\beta\gamma)}$ ,  $K_{zj}^{(\alpha\beta\gamma)}$  ( $j = r, \theta, z$ ) have already been given by eqns (86)-(94). It is convenient to define the following quantities

$$i_{rr}^{(\alpha\beta\gamma)} = I_{rr}^{(\alpha\beta\gamma)} - \rho^{(\alpha\beta\gamma)} \frac{d_\alpha^2}{20} \ddot{W}_{1(200)}^{(\alpha\beta\gamma)} \quad (113)$$

$$i_{r\theta}^{(\alpha\beta\gamma)} = I_{r\theta}^{(\alpha\beta\gamma)} - \rho^{(\alpha\beta\gamma)} \frac{d_\alpha^2}{20} \ddot{W}_{2(200)}^{(\alpha\beta\gamma)} \quad (114)$$

$$i_{rr}^{(\alpha\beta\gamma)} = I_{rr}^{(\alpha\beta\gamma)} - \rho^{(\alpha\beta\gamma)} \frac{d_\alpha^2}{12} \ddot{W}_{1(100)}^{(\alpha\beta\gamma)} \quad (115)$$

$$i_{r\theta}^{(\alpha\beta\gamma)} = I_{r\theta}^{(\alpha\beta\gamma)} - \rho^{(\alpha\beta\gamma)} \frac{d_\alpha^2}{12} \ddot{W}_{2(100)}^{(\alpha\beta\gamma)} \quad (116)$$

$$j_{\theta r}^{(\alpha\beta\gamma)} = J_{\theta r}^{(\alpha\beta\gamma)} - \rho^{(\alpha\beta\gamma)} \frac{h_\beta^2}{20} \ddot{W}_{1(020)}^{(\alpha\beta\gamma)} \quad (117)$$

$$j_{\theta\theta}^{(\alpha\beta\gamma)} = J_{\theta\theta}^{(\alpha\beta\gamma)} - \rho^{(\alpha\beta\gamma)} \frac{h_\beta^2}{20} \ddot{W}_{2(020)}^{(\alpha\beta\gamma)} \quad (118)$$

$$j_{\theta r}^{(\alpha\beta\gamma)} = J_{\theta r}^{(\alpha\beta\gamma)} - \rho^{(\alpha\beta\gamma)} \frac{h_\beta^2}{12} \ddot{W}_{1(010)}^{(\alpha\beta\gamma)} \quad (119)$$

$$j_{\theta\theta}^{(\alpha\beta\gamma)} = J_{\theta\theta}^{(\alpha\beta\gamma)} - \rho^{(\alpha\beta\gamma)} \frac{h_\beta^2}{12} \ddot{W}_{2(010)}^{(\alpha\beta\gamma)} \quad (120)$$

$$k_{zr}^{(\alpha\beta\gamma)} = K_{zr}^{(\alpha\beta\gamma)} - \rho^{(\alpha\beta\gamma)} \frac{l_\gamma^2}{20} \ddot{W}_{1(002)}^{(\alpha\beta\gamma)} \quad (121)$$

$$k_{z\theta}^{(\alpha\beta\gamma)} = K_{z\theta}^{(\alpha\beta\gamma)} - \rho^{(\alpha\beta\gamma)} \frac{l_\gamma^2}{20} \ddot{W}_{2(002)}^{(\alpha\beta\gamma)} \quad (122)$$

$$k_{zz}^{(\alpha\beta\gamma)} = K_{zz}^{(\alpha\beta\gamma)} - \rho^{(\alpha\beta\gamma)} \frac{l_\gamma^2}{12} \ddot{W}_{3(001)}^{(\alpha\beta\gamma)} \quad (123)$$

It should be noted that the above eleven new quantities (113)-(123) are given solely in terms of the stresses  $S_{ij}^{(\alpha\beta\gamma)}$ , eqns (66)-(79), and therefore involve no time derivatives of the unknown displacement coefficients  $W_{i(lmn)}^{(\alpha\beta\gamma)}$ .

**4.11. Volume Averages of the Equations of Motion.** Once the interfacial traction integrals (83)-(85) have been established, we can readily express the volume averages of the three equations of motion, given by eqns (80)-(82), over the  $(\alpha\beta\gamma)$  subcell of the  $(p, q, s)$ th cell. As it can be readily observed, eqn (82) is trivially satisfied, while eqns (80)-(81) are, respectively, given as follows:

$$\begin{aligned} & \rho^{(\alpha\beta\gamma)} [\ddot{W}_{1(000)}^{(\alpha\beta\gamma)} - \frac{d_\alpha^2}{20} \ddot{W}_{1(200)}^{(\alpha\beta\gamma)} - \frac{h_\beta^2}{20} \ddot{W}_{1(020)}^{(\alpha\beta\gamma)} - \frac{l_\gamma^2}{20} \ddot{W}_{1(002)}^{(\alpha\beta\gamma)}] \\ &= i_{rr}^{(\alpha\beta\gamma)} + j_{\theta r}^{(\alpha\beta\gamma)} + k_{zr}^{(\alpha\beta\gamma)} + \frac{1}{R^{(\alpha\beta\gamma)}} [S_{rr}^{(\alpha\beta\gamma)} - S_{\theta\theta}^{(\alpha\beta\gamma)}] \end{aligned} \quad (124)$$

$$\begin{aligned}
& \rho^{(\alpha\beta\gamma)} [\ddot{W}_{2(000)}^{(\alpha\beta\gamma)} - \frac{d_\alpha^2}{20} \ddot{W}_{2(200)}^{(\alpha\beta\gamma)} - \frac{h_\beta^2}{20} \ddot{W}_{2(020)}^{(\alpha\beta\gamma)} - \frac{l_\gamma^2}{20} \ddot{W}_{2(002)}^{(\alpha\beta\gamma)}] \\
&= i_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} + j_{\theta\theta(0,0,0)}^{(\alpha\beta\gamma)} + k_{z\theta(0,0,0)}^{(\alpha\beta\gamma)} + \frac{2}{R^{(\alpha\beta\gamma)}} S_{r\theta(0,0,0)}^{(\alpha\beta\gamma)}
\end{aligned} \tag{125}$$

These two equations form 16 out of the 112 relations needed for the determination of the second time derivatives of the unknown displacement coefficients  $W_{i(lmn)}^{(\alpha\beta\gamma)}$ .

**4.12. Imposition of the Traction Continuity Conditions.** The traction continuity conditions are imposed on an average basis at the subcell and cell interfaces. These conditions imply existence of certain relationships between the aforementioned interfacial traction integrals as described next.

**Interfacial continuity of tractions in the radial direction.** Let us define the following two new quantities:

$$F_{rj}^{(\alpha\beta\gamma)} |^{(p,q,s)} = \sigma_{rj}^{(\alpha\beta\gamma)} |_{\bar{r}(\alpha)=d_\alpha^{(p)}/2}^{(p,q,s)} - \sigma_{rj}^{(\alpha\beta\gamma)} |_{\bar{r}(\alpha)=-d_\alpha^{(p)}/2}^{(p,q,s)} \tag{126}$$

$$G_{rj}^{(\alpha\beta\gamma)} |^{(p,q,s)} = \sigma_{rj}^{(\alpha\beta\gamma)} |_{\bar{r}(\alpha)=d_\alpha^{(p)}/2}^{(p,q,s)} + \sigma_{rj}^{(\alpha\beta\gamma)} |_{\bar{r}(\alpha)=-d_\alpha^{(p)}/2}^{(p,q,s)} \tag{127}$$

with  $j = r, \theta, z$ .

Substituting the interfacial traction continuity conditions (37) and (40) between cells and subcells in the r-direction, we obtain, respectively,

$$F_{rj}^{(1\beta\gamma)} |^{(p,q,s)} = \sigma_{rj}^{(2\beta\gamma)} |_{\bar{r}(2)=-d_2^{(p)}/2}^{(p,q,s)} - \sigma_{rj}^{(2\beta\gamma)} |_{\bar{r}(2)=d_2^{(p-1)}/2}^{(p-1,q,s)} \tag{128}$$

$$G_{rj}^{(1\beta\gamma)} |^{(p,q,s)} = \sigma_{rj}^{(2\beta\gamma)} |_{\bar{r}(2)=-d_2^{(p)}/2}^{(p,q,s)} + \sigma_{rj}^{(2\beta\gamma)} |_{\bar{r}(2)=d_2^{(p-1)}/2}^{(p-1,q,s)} \tag{129}$$

By addition and subtraction of equal quantities to and from the last two equations it can be easily verified that

$$2F_{rj}^{(1\beta\gamma)} |^{(p,q,s)} = [-F_{rj}^{(2\beta\gamma)} + G_{rj}^{(2\beta\gamma)}] |^{(p,q,s)} - [F_{rj}^{(2\beta\gamma)} + G_{rj}^{(2\beta\gamma)}] |^{(p-1,q,s)} \tag{130}$$

$$2G_{rj}^{(1\beta\gamma)} |^{(p,q,s)} = [-F_{rj}^{(2\beta\gamma)} + G_{rj}^{(2\beta\gamma)}] |^{(p,q,s)} + [F_{rj}^{(2\beta\gamma)} + G_{rj}^{(2\beta\gamma)}] |^{(p-1,q,s)} \tag{131}$$

Then employing the definition for  $I_{rj(n,0,0)}^{(\alpha\beta\gamma)}$ , eqn (83), we obtain the corresponding relations:

$$\begin{aligned}
I_{rj(0,0,0)}^{(1\beta\gamma)} |^{(p,q,s)} &= -Z_1 I_{rj(0,0,0)}^{(2\beta\gamma)} |^{(p,q,s)} - Z_2 I_{rj(0,0,0)}^{(2\beta\gamma)} |^{(p-1,q,s)} \\
&+ Z_3 I_{rj(1,0,0)}^{(2\beta\gamma)} |^{(p,q,s)} - Z_4 I_{rj(1,0,0)}^{(2\beta\gamma)} |^{(p-1,q,s)}
\end{aligned} \tag{132}$$

$$\begin{aligned}
I_{rj(1,0,0)}^{(1\beta\gamma)} |^{(p,q,s)} &= Z_5 I_{rj(1,0,0)}^{(2\beta\gamma)} |^{(p,q,s)} + Z_6 I_{rj(1,0,0)}^{(2\beta\gamma)} |^{(p-1,q,s)} \\
&- Z_7 I_{rj(0,0,0)}^{(2\beta\gamma)} |^{(p,q,s)} + Z_8 I_{rj(0,0,0)}^{(2\beta\gamma)} |^{(p-1,q,s)}
\end{aligned} \tag{133}$$

where  $j = r, \theta$ , since for  $j = z$  all quantities vanish.

The above equations (132)-(133) are applicable for internal cells where  $p = 2, 3, \dots, N_p$ . For the 1st cell  $p = 1$  the continuity of tractions at the interfaces between the subcells of this cell can be easily shown to yield the single relation:

$$[G_{rj}^{(1\beta\gamma)} + F_{rj}^{(1\beta\gamma)}]^{(1,q,s)} = [G_{rj}^{(2\beta\gamma)} - F_{rj}^{(2\beta\gamma)}]^{(1,q,s)} \quad (134)$$

This single relation replaces the above corresponding two relations (132)-(133) at interior cells. This relation provides

$$[I_{rj(0,0,0)}^{(1\beta\gamma)} + \frac{2}{d_1} I_{rj(1,0,0)}^{(1\beta\gamma)}]^{(1,q,s)} = 2[-Z_1 I_{rj(0,0,0)}^{(2\beta\gamma)} + Z_3 I_{rj(1,0,0)}^{(2\beta\gamma)}]^{(1,q,s)} \quad (135)$$

where  $j = r, \theta$ , since for  $j = z$  this equation is identically satisfied. Again, this relation replaces the above two equations that are applicable at interior cells.

Substituting the established expressions for the interfacial traction integrals into relations (132)-(133), we obtain the following equations for internal cells when  $j = r$

$$\begin{aligned} -[\rho^{(1\beta\gamma)} \frac{d_1^2}{20} \ddot{W}_{1(200)}^{(1\beta\gamma)}]^{(p,q,s)} & - Z_1 [\rho^{(2\beta\gamma)} \frac{d_2^2}{20} \ddot{W}_{1(200)}^{(2\beta\gamma)}]^{(p,q,s)} - Z_2 [\rho^{(2\beta\gamma)} \frac{d_2^2}{20} \ddot{W}_{1(200)}^{(2\beta\gamma)}]^{(p-1,q,s)} \\ & + Z_3 [\rho^{(2\beta\gamma)} \frac{d_2^2}{12} \ddot{W}_{1(100)}^{(2\beta\gamma)}]^{(p,q,s)} - Z_4 [\rho^{(2\beta\gamma)} \frac{d_2^2}{12} \ddot{W}_{1(100)}^{(2\beta\gamma)}]^{(p-1,q,s)} \\ & = i_{rr(0,0,0)}^{(1\beta\gamma)} |^{(p,q,s)} + Z_1 i_{rr(0,0,0)}^{(2\beta\gamma)} |^{(p,q,s)} + Z_2 i_{rr(0,0,0)}^{(2\beta\gamma)} |^{(p-1,q,s)} \\ & - Z_3 i_{rr(1,0,0)}^{(2\beta\gamma)} |^{(p,q,s)} + Z_4 i_{rr(1,0,0)}^{(2\beta\gamma)} |^{(p-1,q,s)} \end{aligned} \quad (136)$$

$$\begin{aligned} -[\rho^{(1\beta\gamma)} \frac{d_1^2}{12} \ddot{W}_{1(100)}^{(1\beta\gamma)}]^{(p,q,s)} & + Z_5 [\rho^{(2\beta\gamma)} \frac{d_2^2}{12} \ddot{W}_{1(100)}^{(2\beta\gamma)}]^{(p,q,s)} + Z_6 [\rho^{(2\beta\gamma)} \frac{d_2^2}{12} \ddot{W}_{1(100)}^{(2\beta\gamma)}]^{(p-1,q,s)} \\ & - Z_7 [\rho^{(2\beta\gamma)} \frac{d_2^2}{20} \ddot{W}_{1(200)}^{(2\beta\gamma)}]^{(p,q,s)} + Z_8 [\rho^{(2\beta\gamma)} \frac{d_2^2}{20} \ddot{W}_{1(200)}^{(2\beta\gamma)}]^{(p-1,q,s)} \\ & = i_{rr(1,0,0)}^{(1\beta\gamma)} |^{(p,q,s)} - Z_5 i_{rr(1,0,0)}^{(2\beta\gamma)} |^{(p,q,s)} - Z_6 i_{rr(1,0,0)}^{(2\beta\gamma)} |^{(p-1,q,s)} \\ & + Z_7 i_{rr(0,0,0)}^{(2\beta\gamma)} |^{(p,q,s)} - Z_8 i_{rr(0,0,0)}^{(2\beta\gamma)} |^{(p-1,q,s)} \end{aligned} \quad (137)$$

For  $j = \theta$ , the two relations (132)-(133) become:

$$\begin{aligned} -[\rho^{(1\beta\gamma)} \frac{d_1^2}{20} \ddot{W}_{2(200)}^{(1\beta\gamma)}]^{(p,q,s)} & - Z_1 [\rho^{(2\beta\gamma)} \frac{d_2^2}{20} \ddot{W}_{2(200)}^{(2\beta\gamma)}]^{(p,q,s)} - Z_2 [\rho^{(2\beta\gamma)} \frac{d_2^2}{20} \ddot{W}_{2(200)}^{(2\beta\gamma)}]^{(p-1,q,s)} \\ & + Z_3 [\rho^{(2\beta\gamma)} \frac{d_2^2}{12} \ddot{W}_{2(100)}^{(2\beta\gamma)}]^{(p,q,s)} - Z_4 [\rho^{(2\beta\gamma)} \frac{d_2^2}{12} \ddot{W}_{2(100)}^{(2\beta\gamma)}]^{(p-1,q,s)} \\ & = i_{r\theta(0,0,0)}^{(1\beta\gamma)} |^{(p,q,s)} + Z_1 i_{r\theta(0,0,0)}^{(2\beta\gamma)} |^{(p,q,s)} + Z_2 i_{r\theta(0,0,0)}^{(2\beta\gamma)} |^{(p-1,q,s)} \\ & - Z_3 i_{r\theta(1,0,0)}^{(2\beta\gamma)} |^{(p,q,s)} + Z_4 i_{r\theta(1,0,0)}^{(2\beta\gamma)} |^{(p-1,q,s)} \end{aligned} \quad (138)$$

$$\begin{aligned} -[\rho^{(1\beta\gamma)} \frac{d_1^2}{12} \ddot{W}_{2(100)}^{(1\beta\gamma)}]^{(p,q,s)} & + Z_5 [\rho^{(2\beta\gamma)} \frac{d_2^2}{12} \ddot{W}_{2(100)}^{(2\beta\gamma)}]^{(p,q,s)} + Z_6 [\rho^{(2\beta\gamma)} \frac{d_2^2}{12} \ddot{W}_{2(100)}^{(2\beta\gamma)}]^{(p-1,q,s)} \\ & - Z_7 [\rho^{(2\beta\gamma)} \frac{d_2^2}{20} \ddot{W}_{2(200)}^{(2\beta\gamma)}]^{(p,q,s)} + Z_8 [\rho^{(2\beta\gamma)} \frac{d_2^2}{20} \ddot{W}_{2(200)}^{(2\beta\gamma)}]^{(p-1,q,s)} \\ & = i_{r\theta(1,0,0)}^{(1\beta\gamma)} |^{(p,q,s)} - Z_5 i_{r\theta(1,0,0)}^{(2\beta\gamma)} |^{(p,q,s)} - Z_6 i_{r\theta(1,0,0)}^{(2\beta\gamma)} |^{(p-1,q,s)} \\ & + Z_7 i_{r\theta(0,0,0)}^{(2\beta\gamma)} |^{(p,q,s)} - Z_8 i_{r\theta(0,0,0)}^{(2\beta\gamma)} |^{(p-1,q,s)} \end{aligned} \quad (139)$$

Consequently, these four equations (136)-(139) provide additional 16 relations for the second time derivatives of the unknown displacement coefficients  $W_{i(lmn)}^{(\alpha\beta\gamma)}$ .

The two relations (132)-(133) are applicable at internal cells  $p = 2, \dots, N_p$ . For the boundary cell  $p = 1$  we use the corresponding relations (135). For  $j = r$  we obtain

$$\begin{aligned} & [\rho^{(1\beta\gamma)} \frac{d_1^2}{20} \ddot{W}_{1(200)}^{(1\beta\gamma)} + 2Z_1 \rho^{(2\beta\gamma)} \frac{d_2^2}{20} \ddot{W}_{1(200)}^{(2\beta\gamma)} + \rho^{(1\beta\gamma)} \frac{d_1}{6} \ddot{W}_{1(100)}^{(1\beta\gamma)} \\ & - Z_3 \rho^{(2\beta\gamma)} \frac{d_2^2}{6} \ddot{W}_{1(100)}^{(2\beta\gamma)}]^{(1,q,s)} = [-i_{rr(0,0,0)}^{(1\beta\gamma)} \\ & - 2Z_1 i_{rr(0,0,0)}^{(2\beta\gamma)} - \frac{2}{d_1} i_{rr(1,0,0)}^{(1\beta\gamma)} + 2Z_3 i_{rr(1,0,0)}^{(2\beta\gamma)}]^{(1,q,s)} \end{aligned} \quad (140)$$

while for  $j = \theta$ :

$$\begin{aligned} & [\rho^{(1\beta\gamma)} \frac{d_1^2}{20} \ddot{W}_{2(200)}^{(1\beta\gamma)} + 2Z_1 \rho^{(2\beta\gamma)} \frac{d_2^2}{20} \ddot{W}_{2(200)}^{(2\beta\gamma)} + \rho^{(1\beta\gamma)} \frac{d_1}{6} \ddot{W}_{2(100)}^{(1\beta\gamma)} \\ & - Z_3 \rho^{(2\beta\gamma)} \frac{d_2^2}{6} \ddot{W}_{2(100)}^{(2\beta\gamma)}]^{(1,q,s)} = [-i_{r\theta(0,0,0)}^{(1\beta\gamma)} \\ & - 2Z_1 i_{r\theta(0,0,0)}^{(2\beta\gamma)} - \frac{2}{d_1} i_{r\theta(1,0,0)}^{(1\beta\gamma)} + 2Z_3 i_{r\theta(1,0,0)}^{(2\beta\gamma)}]^{(1,q,s)} \end{aligned} \quad (141)$$

These are 8 relations that are applicable for the boundary cell  $p = 1$ . For this cell however there are another 8 relations that express the applied boundary conditions at the surface  $r = R_0$ . These latter relations will be presented in Section 4.14. Consequently, we have altogether 16 relations for the second time derivatives of the unknown displacement coefficients  $W_{i(lmn)}^{(\alpha\beta\gamma)}$  in this boundary cell,  $p = 1$ , (just like any inner cell  $p \neq 1$ ).

**Interfacial continuity of tractions in the angular direction.** A similar analysis for the traction continuity conditions in the  $\theta$ -direction yields for internal cells  $q = 2, 3, \dots, N_q$

$$\begin{aligned} J_{\theta j(0,0,0)}^{(\alpha 1\gamma)} \Big|^{(p,q,s)} &= -Y_1 J_{\theta j(0,0,0)}^{(\alpha 2\gamma)} \Big|^{(p,q,s)} - Y_2 J_{\theta j(0,0,0)}^{(\alpha 2\gamma)} \Big|^{(p,q-1,s)} \\ &+ Y_3 J_{\theta j(0,1,0)}^{(\alpha 2\gamma)} \Big|^{(p,q,s)} - Y_4 J_{\theta j(0,1,0)}^{(\alpha 2\gamma)} \Big|^{(p,q-1,s)} \end{aligned} \quad (142)$$

$$\begin{aligned} J_{\theta j(0,1,0)}^{(\alpha 1\gamma)} \Big|^{(p,q,s)} &= Y_5 J_{\theta j(0,1,0)}^{(\alpha 2\gamma)} \Big|^{(p,q,s)} + Y_6 J_{\theta j(0,1,0)}^{(\alpha 2\gamma)} \Big|^{(p,q-1,s)} \\ &- Y_7 J_{\theta j(0,0,0)}^{(\alpha 2\gamma)} \Big|^{(p,q,s)} + Y_8 J_{\theta j(0,0,0)}^{(\alpha 2\gamma)} \Big|^{(p,q-1,s)} \end{aligned} \quad (143)$$

where  $j = r, \theta$ , since for  $j = z$  all quantities vanish. The above equations (142)-(143) are applicable for internal cells where  $q = 2, 3, \dots, N_q$ .

For the 1st cell  $q = 1$ , the continuity of tractions at the interfaces between the subcells of this cell yields the single relation:

$$\left[ J_{\theta j(0,0,0)}^{(\alpha 1\gamma)} + \frac{2}{h_1} J_{\theta j(0,1,0)}^{(\alpha 1\gamma)} \right]^{(p,1,s)} = 2[-Y_1 J_{\theta j(0,0,0)}^{(\alpha 2\gamma)} + Y_3 J_{\theta j(0,1,0)}^{(\alpha 2\gamma)}]^{(p,1,s)} \quad (144)$$

where  $j = r, \theta$ , since for  $j = z$  this equation is trivially satisfied.

Substituting in (142)-143) the established expressions for the interfacial traction integrals we readily obtain for internal cells the following expressions when  $j = r$ :

$$\begin{aligned}
& -[\rho^{(\alpha 1 \gamma)} \frac{h_1^2}{20} \ddot{W}_{1(020)}^{(\alpha 1 \gamma)}]_{(p, q, s)} - Y_1[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{20} \ddot{W}_{1(020)}^{(\alpha 2 \gamma)}]_{(p, q, s)} - Y_2[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{20} \ddot{W}_{1(020)}^{(\alpha 2 \gamma)}]_{(p, q-1, s)} \\
& + Y_3[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{12} \ddot{W}_{1(010)}^{(\alpha 2 \gamma)}]_{(p, q, s)} - Y_4[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{12} \ddot{W}_{1(010)}^{(\alpha 2 \gamma)}]_{(p, q-1, s)} \\
& = j_{\theta r(0,0,0)}^{(\alpha 1 \gamma)} |_{(p, q, s)} + Y_1 j_{\theta r(0,0,0)}^{(\alpha 2 \gamma)} |_{(p, q, s)} + Y_2 j_{\theta r(0,0,0)}^{(\alpha 2 \gamma)} |_{(p, q-1, s)} \\
& - Y_3 j_{\theta r(0,1,0)}^{(\alpha 2 \gamma)} |_{(p, q, s)} + Y_4 j_{\theta r(0,1,0)}^{(\alpha 2 \gamma)} |_{(p, q-1, s)} \tag{145}
\end{aligned}$$

$$\begin{aligned}
& -[\rho^{(\alpha 1 \gamma)} \frac{h_1^2}{12} \ddot{W}_{1(010)}^{(\alpha 1 \gamma)}]_{(p, q, s)} + Y_5[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{12} \ddot{W}_{1(010)}^{(\alpha 2 \gamma)}]_{(p, q, s)} + Y_6[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{12} \ddot{W}_{1(010)}^{(\alpha 2 \gamma)}]_{(p, q-1, s)} \\
& - Y_7[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{20} \ddot{W}_{1(020)}^{(\alpha 2 \gamma)}]_{(p, q, s)} + Y_8[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{20} \ddot{W}_{1(020)}^{(\alpha 2 \gamma)}]_{(p, q-1, s)} \\
& = j_{\theta r(0,1,0)}^{(\alpha 1 \gamma)} |_{(p, q, s)} - Y_5 j_{\theta r(0,1,0)}^{(\alpha 2 \gamma)} |_{(p, q, s)} - Y_6 j_{\theta r(0,1,0)}^{(\alpha 2 \gamma)} |_{(p, q-1, s)} \\
& + Y_7 j_{\theta r(0,0,0)}^{(\alpha 2 \gamma)} |_{(p, q, s)} - Y_8 j_{\theta r(0,0,0)}^{(\alpha 2 \gamma)} |_{(p, q-1, s)} \tag{146}
\end{aligned}$$

For  $j = \theta$  the two relations (142)-(143) yield,

$$\begin{aligned}
& -[\rho^{(\alpha 1 \gamma)} \frac{h_1^2}{20} \ddot{W}_{2(020)}^{(\alpha 1 \gamma)}]_{(p, q, s)} - Y_1[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{20} \ddot{W}_{2(020)}^{(\alpha 2 \gamma)}]_{(p, q, s)} - Y_2[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{20} \ddot{W}_{2(020)}^{(\alpha 2 \gamma)}]_{(p, q-1, s)} \\
& + Y_3[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{12} \ddot{W}_{2(010)}^{(\alpha 2 \gamma)}]_{(p, q, s)} - Y_4[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{12} \ddot{W}_{2(010)}^{(\alpha 2 \gamma)}]_{(p, q-1, s)} \\
& = j_{\theta \theta(0,0,0)}^{(\alpha 1 \gamma)} |_{(p, q, s)} + Y_1 j_{\theta \theta(0,0,0)}^{(\alpha 2 \gamma)} |_{(p, q, s)} + Y_2 j_{\theta \theta(0,0,0)}^{(\alpha 2 \gamma)} |_{(p, q-1, s)} \\
& - Y_3 j_{\theta \theta(0,1,0)}^{(\alpha 2 \gamma)} |_{(p, q, s)} + Y_4 j_{\theta \theta(0,1,0)}^{(\alpha 2 \gamma)} |_{(p, q-1, s)} \tag{147}
\end{aligned}$$

$$\begin{aligned}
& -[\rho^{(\alpha 1 \gamma)} \frac{h_1^2}{12} \ddot{W}_{2(010)}^{(\alpha 1 \gamma)}]_{(p, q, s)} + Y_5[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{12} \ddot{W}_{2(010)}^{(\alpha 2 \gamma)}]_{(p, q, s)} + Y_6[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{12} \ddot{W}_{2(010)}^{(\alpha 2 \gamma)}]_{(p, q-1, s)} \\
& - Y_7[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{20} \ddot{W}_{2(020)}^{(\alpha 2 \gamma)}]_{(p, q, s)} + Y_8[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{20} \ddot{W}_{2(020)}^{(\alpha 2 \gamma)}]_{(p, q-1, s)} \\
& = j_{\theta \theta(0,1,0)}^{(\alpha 1 \gamma)} |_{(p, q, s)} - Y_5 j_{\theta \theta(0,1,0)}^{(\alpha 2 \gamma)} |_{(p, q, s)} - Y_6 j_{\theta \theta(0,1,0)}^{(\alpha 2 \gamma)} |_{(p, q-1, s)} \\
& + Y_7 j_{\theta \theta(0,0,0)}^{(\alpha 2 \gamma)} |_{(p, q, s)} - Y_8 j_{\theta \theta(0,0,0)}^{(\alpha 2 \gamma)} |_{(p, q-1, s)} \tag{148}
\end{aligned}$$

Consequently, the conditions that the tractions are continuous at the interfaces in the  $\theta$  direction provide 16 additional relations which are valid for internal cells  $q = 2, 3, \dots, N_q$ .

For the boundary cells  $q = 1$ , we obtain from eqn (144) the following expression when  $j = r$ :

$$\begin{aligned}
& [\rho^{(\alpha 1 \gamma)} \frac{h_1^2}{20} \ddot{W}_{1(020)}^{(\alpha 1 \gamma)}] + 2Y_1 \rho^{(\alpha 2 \gamma)} \frac{h_2^2}{20} \ddot{W}_{1(020)}^{(\alpha 2 \gamma)} + \rho^{(\alpha 1 \gamma)} \frac{h_1}{6} \ddot{W}_{1(010)}^{(\alpha 1 \gamma)} \\
& - Y_3 \rho^{(\alpha 2 \gamma)} \frac{h_2^2}{6} \ddot{W}_{1(010)}^{(\alpha 2 \gamma)}]_{(p, 1, s)} = [-j_{\theta r(0,0,0)}^{(\alpha 1 \gamma)} \\
& - 2Y_1 j_{\theta r(0,0,0)}^{(\alpha 2 \gamma)} - \frac{2}{h_1} j_{\theta r(0,1,0)}^{(\alpha 1 \gamma)} + 2Y_3 j_{\theta r(0,1,0)}^{(\alpha 2 \gamma)}]_{(p, 1, s)} \tag{149}
\end{aligned}$$

while for  $j = \theta$ :

$$\begin{aligned}
& [\rho^{(\alpha 1 \gamma)} \frac{h_1^2}{20} \ddot{W}_{2(020)}^{(\alpha 1 \gamma)} + 2Y_1 \rho^{(\alpha 2 \gamma)} \frac{h_2^2}{20} \ddot{W}_{2(020)}^{(\alpha 2 \gamma)} + \rho^{(\alpha 1 \gamma)} \frac{h_1}{6} \ddot{W}_{2(010)}^{(\alpha 1 \gamma)} \\
& - Y_3 \rho^{(\alpha 2 \gamma)} \frac{h_2^2}{6} \ddot{W}_{2(010)}^{(\alpha 2 \gamma)}]^{(p,1,s)} = [-j_{\theta\theta(0,0,0)}^{(\alpha 1 \gamma)} \\
& - 2Y_1 j_{\theta\theta(0,0,0)}^{(\alpha 2 \gamma)} - \frac{2}{h_1} j_{\theta\theta(0,1,0)}^{(\alpha 1 \gamma)} + 2Y_3 j_{\theta\theta(0,1,0)}^{(\alpha 2 \gamma)}]^{(p,1,s)} \quad (150)
\end{aligned}$$

Thus for the boundary cells  $q = 1$  we have 8 relations which together with the 8 imposed boundary conditions at the surface of this cell  $\theta = 0$  (which will be discussed in Section 4.14) form the required 16 relations (just like any internal cell  $q \neq 1$ ).

**Interfacial continuity of tractions in the axial direction.** Let us consider the traction continuity conditions in the periodic axial  $z$ -direction. Here we define the two quantities

$$F_{zj}^{(\alpha\beta\gamma)} |_{(p,q,s)} = \sigma_{zj}^{(\alpha\beta\gamma)} |_{\bar{z}(\gamma)=l_\gamma/2} - \sigma_{zj}^{(\alpha\beta\gamma)} |_{\bar{z}(\gamma)=-l_\gamma/2} \quad (151)$$

$$G_{zj}^{(\alpha\beta\gamma)} |_{(p,q,s)} = \sigma_{zj}^{(\alpha\beta\gamma)} |_{\bar{z}(\gamma)=l_\gamma/2} + \sigma_{zj}^{(\alpha\beta\gamma)} |_{\bar{z}(\gamma)=-l_\gamma/2} \quad (152)$$

with  $j = r, \theta, z$ . It can be readily established that due to periodicity of the stresses between repeating cells in the axial direction, the continuity of tractions at the interfaces between the subcells of the cell yields

$$F_{zj}^{(\alpha\beta 1)} |_{(p,q,s)} = -F_{zj}^{(\alpha\beta 2)} |_{(p,q,s)} \quad (153)$$

$$G_{zj}^{(\alpha\beta 1)} |_{(p,q,s)} = G_{zj}^{(\alpha\beta 2)} |_{(p,q,s)} \quad (154)$$

The above first equation (153) gives the following nontrivial relations

$$[l_1 K_{zj(0,0,0)}^{(\alpha\beta 1)} + l_2 K_{zj(0,0,0)}^{(\alpha\beta 2)}]^{(p,q,s)} = 0 \quad (155)$$

for  $j = r$  and  $\theta$ , while for  $j = z$  we obtain from eqn (154)

$$[K_{zz(0,0,1)}^{(\alpha\beta 1)} - K_{zz(0,0,1)}^{(\alpha\beta 2)}]^{(p,q,s)} = 0 \quad (156)$$

Consequently, the following relations can be readily established in the cell  $(p, q, s)$

$$\rho^{(\alpha\beta 1)} \frac{l_1^3}{20} \ddot{W}_{1(002)}^{(\alpha\beta 1)} + \rho^{(\alpha\beta 2)} \frac{l_2^3}{20} \ddot{W}_{1(002)}^{(\alpha\beta 2)} = -l_1 k_{zr(0,0,0)}^{(\alpha\beta 1)} - l_2 k_{zr(0,0,0)}^{(\alpha\beta 2)} \quad (157)$$

for  $j = r$ , and

$$\rho^{(\alpha\beta 1)} \frac{l_1^3}{20} \ddot{W}_{2(002)}^{(\alpha\beta 1)} + \rho^{(\alpha\beta 2)} \frac{l_2^3}{20} \ddot{W}_{2(002)}^{(\alpha\beta 2)} = -l_1 k_{z\theta(0,0,0)}^{(\alpha\beta 1)} - l_2 k_{z\theta(0,0,0)}^{(\alpha\beta 2)} \quad (158)$$

for  $j = \theta$ . In addition we obtain for  $j = z$

$$\rho^{(\alpha\beta 1)} \frac{l_1^2}{12} \ddot{W}_{3(001)}^{(\alpha\beta 1)} - \rho^{(\alpha\beta 2)} \frac{l_2^2}{12} \ddot{W}_{3(001)}^{(\alpha\beta 2)} = -k_{zz(0,0,1)}^{(\alpha\beta 1)} + k_{zz(0,0,1)}^{(\alpha\beta 2)} \quad (159)$$

These relations provide 12 additional equations in the 2nd derivatives  $\ddot{W}_{j(lmn)}^{(\alpha\beta\gamma)}$ .

In summary, the imposition of traction continuity between neighboring cells and between the subcells of the cell in the radial, angular and axial directions provides a total of 44 equations in the 2nd time derivatives  $\ddot{W}_{j(lmn)}^{(\alpha\beta\gamma)}$ .

#### 4.13. Imposition of the Displacement Continuity Conditions.

**Continuity of displacements in the radial direction.** The continuity conditions, imposed in the average sense, on the displacement vector at the inner surfaces as well as between neighboring cells in the  $r$ -direction, eqns (43) and (46), yield

$$[\ddot{W}_{1(000)}^{(1\beta\gamma)} + \frac{d_1}{2}\ddot{W}_{1(100)}^{(1\beta\gamma)} + \frac{d_1^2}{4}\ddot{W}_{1(200)}^{(1\beta\gamma)} - \ddot{W}_{1(000)}^{(2\beta\gamma)} + \frac{d_2}{2}\ddot{W}_{1(100)}^{(2\beta\gamma)} - \frac{d_2^2}{4}\ddot{W}_{1(200)}^{(2\beta\gamma)}]^{(p,q,s)} = 0 \quad (160)$$

$$[\ddot{W}_{1(000)}^{(2\beta\gamma)} + \frac{d_2}{2}\ddot{W}_{1(100)}^{(2\beta\gamma)} + \frac{d_2^2}{4}\ddot{W}_{1(200)}^{(2\beta\gamma)}]^{(p,q,s)} = [\ddot{W}_{1(000)}^{(1\beta\gamma)} - \frac{d_1}{2}\ddot{W}_{1(100)}^{(1\beta\gamma)} + \frac{d_1^2}{4}\ddot{W}_{1(200)}^{(1\beta\gamma)}]^{(p+1,q,s)} \quad (161)$$

$$[\ddot{W}_{2(000)}^{(1\beta\gamma)} + \frac{d_1}{2}\ddot{W}_{2(100)}^{(1\beta\gamma)} + \frac{d_1^2}{4}\ddot{W}_{2(200)}^{(1\beta\gamma)} - \ddot{W}_{2(000)}^{(2\beta\gamma)} + \frac{d_2}{2}\ddot{W}_{2(100)}^{(2\beta\gamma)} - \frac{d_2^2}{4}\ddot{W}_{2(200)}^{(2\beta\gamma)}]^{(p,q,s)} = 0 \quad (162)$$

$$[\ddot{W}_{2(000)}^{(2\beta\gamma)} + \frac{d_2}{2}\ddot{W}_{2(100)}^{(2\beta\gamma)} + \frac{d_2^2}{4}\ddot{W}_{2(200)}^{(2\beta\gamma)}]^{(p,q,s)} = [\ddot{W}_{2(000)}^{(1\beta\gamma)} - \frac{d_1}{2}\ddot{W}_{2(100)}^{(1\beta\gamma)} + \frac{d_1^2}{4}\ddot{W}_{2(200)}^{(1\beta\gamma)}]^{(p+1,q,s)} \quad (163)$$

Obviously, eqns (161) and (163), are not applicable at the boundary cell  $p = N_p$ . These relations are replaced by the boundary conditions that are prescribed at the surface  $r = R_1$  of this cell as will be discussed in Section 4.14.

**Continuity of displacements in the angular direction.** The continuity conditions, imposed in the average sense, on the displacement vector, at the inner surfaces as well as between neighboring cells in the  $\theta$ -direction, eqns (44) and (47), yield

$$[\ddot{W}_{1(000)}^{(\alpha 1\gamma)} + \frac{h_1}{2}\ddot{W}_{1(010)}^{(\alpha 1\gamma)} + \frac{h_1^2}{4}\ddot{W}_{1(020)}^{(\alpha 1\gamma)} - \ddot{W}_{1(000)}^{(\alpha 2\gamma)} + \frac{h_2}{2}\ddot{W}_{1(010)}^{(\alpha 2\gamma)} - \frac{h_2^2}{4}\ddot{W}_{1(020)}^{(\alpha 2\gamma)}]^{(p,q,s)} = 0 \quad (164)$$

$$[\ddot{W}_{1(000)}^{(\alpha 2\gamma)} + \frac{h_2}{2}\ddot{W}_{1(010)}^{(\alpha 2\gamma)} + \frac{h_2^2}{4}\ddot{W}_{1(020)}^{(\alpha 2\gamma)}]^{(p,q,s)} = [\ddot{W}_{1(000)}^{(\alpha 1\gamma)} - \frac{h_1}{2}\ddot{W}_{1(010)}^{(\alpha 1\gamma)} + \frac{h_1^2}{4}\ddot{W}_{1(020)}^{(\alpha 1\gamma)}]^{(p,q+1,s)} \quad (165)$$

$$[\ddot{W}_{2(000)}^{(\alpha 1\gamma)} + \frac{h_1}{2}\ddot{W}_{2(010)}^{(\alpha 1\gamma)} + \frac{h_1^2}{4}\ddot{W}_{2(020)}^{(\alpha 1\gamma)} - \ddot{W}_{2(000)}^{(\alpha 2\gamma)} + \frac{h_2}{2}\ddot{W}_{2(010)}^{(\alpha 2\gamma)} - \frac{h_2^2}{4}\ddot{W}_{2(020)}^{(\alpha 2\gamma)}]^{(p,q,s)} = 0 \quad (166)$$

$$[\ddot{W}_{2(000)}^{(\alpha 2\gamma)} + \frac{h_2}{2}\ddot{W}_{2(010)}^{(\alpha 2\gamma)} + \frac{h_2^2}{4}\ddot{W}_{2(020)}^{(\alpha 2\gamma)}]^{(p,q,s)} = [\ddot{W}_{2(000)}^{(\alpha 1\gamma)} - \frac{h_1}{2}\ddot{W}_{2(010)}^{(\alpha 1\gamma)} + \frac{h_1^2}{4}\ddot{W}_{2(020)}^{(\alpha 1\gamma)}]^{(p,q+1,s)} \quad (167)$$

Equations (165) and (167) are not applicable at the boundary cell  $q = N_q$ . These relations are replaced by the boundary conditions that are prescribed at the surface  $\theta = \Theta$  of this cell as will be discussed in Section 4.14.

**Continuity of displacements in the axial direction.** The continuity conditions, imposed in the average sense, on the displacement vector in the periodic  $z$ -direction, eqns (45), provide

$$[\ddot{W}_{1(000)}^{(\alpha\beta 1)} + \frac{l_1^2}{4}\ddot{W}_{1(002)}^{(\alpha\beta 1)} - \ddot{W}_{1(000)}^{(\alpha\beta 2)} - \frac{l_2^2}{4}\ddot{W}_{1(002)}^{(\alpha\beta 2)}]^{(p,q,s)} = 0 \quad (168)$$

$$[\ddot{W}_{2(000)}^{(\alpha\beta 1)} + \frac{l_1^2}{4}\ddot{W}_{2(002)}^{(\alpha\beta 1)} - \ddot{W}_{2(000)}^{(\alpha\beta 2)} - \frac{l_2^2}{4}\ddot{W}_{2(002)}^{(\alpha\beta 2)}]^{(p,q,s)} = 0 \quad (169)$$

Furthermore,

$$[l_1\ddot{W}_{3(001)}^{(\alpha\beta 1)} + l_2\ddot{W}_{3(001)}^{(\alpha\beta 2)}]^{(p,q,s)} = (l_1 + l_2)\ddot{\epsilon}_{zz} \quad (170)$$

where  $\bar{\epsilon}_{zz}$  is the global axial strain that will be discussed in Section 4.15. This last relation follows from a homogenization procedure in the periodic direction that has been presented in Ref. [19].

These displacement continuity conditions provide altogether 44 relations in the 2nd time derivatives  $\ddot{W}_j^{(\alpha\beta\gamma)}$ .

**4.14. Imposition of the Boundary Conditions.** The tractions at the bottom ( $r = R_0$ ) and top ( $r = R_1$ ) surfaces are imposed by employing the relations:

$$G_{rj}^{(1\beta\gamma)}|_{(1,q,s)} - F_{rj}^{(1\beta\gamma)}|_{(1,q,s)} = 2\sigma_{rj}^{(1\beta\gamma)}|_{\bar{r}(1)=-d_1^{(1)}/2}^{(1,q,s)} \quad (171)$$

$$G_{rj}^{(2\beta\gamma)}|_{(N_p,q,s)} + F_{rj}^{(2\beta\gamma)}|_{(N_p,q,s)} = 2\sigma_{rj}^{(2\beta\gamma)}|_{\bar{r}(2)=d_2^{(N_p)}/2}^{(N_p,q,s)} \quad (172)$$

where  $j = r, \theta$ . By averaging these relations over the surfaces upon which these tractions are imposed we obtain, respectively,

$$[I_{rj(1,0,0)}^{(1\beta\gamma)} - \frac{d_1}{2} I_{rj(0,0,0)}^{(1\beta\gamma)}]^{(1,q,s)} = f_{Bj}(\theta, t) \quad (173)$$

$$[I_{rj(1,0,0)}^{(2\beta\gamma)} + \frac{d_2}{2} I_{rj(0,0,0)}^{(2\beta\gamma)}]^{(N_p,q,s)} = f_{Tj}(\theta, t) \quad (174)$$

It follows, by employing the established expressions for the interfacial traction integrals  $I_{rj(l,m,n)}^{(\alpha\beta\gamma)}$ , that the following equations are obtained in terms of the unknown 2nd time derivatives of the coefficients  $W_j^{(\alpha\beta\gamma)}$ :

$$[\rho^{(1\beta\gamma)} \frac{d_1^2}{12} \ddot{W}_{1(100)}^{(1\beta\gamma)} - \rho^{(1\beta\gamma)} \frac{d_1^3}{40} \ddot{W}_{1(200)}^{(1\beta\gamma)}]^{(1,q,s)} = [-i_{rr(1,0,0)}^{(1\beta\gamma)} + \frac{d_1}{2} i_{rr(0,0,0)}^{(1\beta\gamma)}]^{(1,q,s)} + f_{Br}(\theta, t) \quad (175)$$

$$[\rho^{(1\beta\gamma)} \frac{d_1^2}{12} \ddot{W}_{2(100)}^{(1\beta\gamma)} - \rho^{(1\beta\gamma)} \frac{d_1^3}{40} \ddot{W}_{2(200)}^{(1\beta\gamma)}]^{(1,q,s)} = [-i_{r\theta(1,0,0)}^{(1\beta\gamma)} + \frac{d_1}{2} i_{r\theta(0,0,0)}^{(1\beta\gamma)}]^{(1,q,s)} + f_{B\theta}(\theta, t) \quad (176)$$

$$[\rho^{(2\beta\gamma)} \frac{d_2^2}{12} \ddot{W}_{1(100)}^{(2\beta\gamma)} + \rho^{(2\beta\gamma)} \frac{d_2^3}{40} \ddot{W}_{1(200)}^{(2\beta\gamma)}]^{(N_p,q,s)} = [-i_{rr(1,0,0)}^{(2\beta\gamma)} - \frac{d_2}{2} i_{rr(0,0,0)}^{(2\beta\gamma)}]^{(N_p,q,s)} + f_{Tr}(\theta, t) \quad (177)$$

$$[\rho^{(2\beta\gamma)} \frac{d_2^2}{12} \ddot{W}_{2(100)}^{(2\beta\gamma)} + \rho^{(2\beta\gamma)} \frac{d_2^3}{40} \ddot{W}_{2(200)}^{(2\beta\gamma)}]^{(N_p,q,s)} = [-i_{r\theta(1,0,0)}^{(2\beta\gamma)} - \frac{d_2}{2} i_{r\theta(0,0,0)}^{(2\beta\gamma)}]^{(N_p,q,s)} + f_{T\theta}(\theta, t) \quad (178)$$

Similar analysis provides the following relations for the tractions imposed in the average sense at the left ( $\theta = 0$ ) and right ( $\theta = \Theta$ ) boundaries:

$$[J_{\theta j(0,1,0)}^{(\alpha 1\gamma)} - \frac{h_1}{2} J_{\theta j(0,0,0)}^{(\alpha 1\gamma)}]^{(p,1,s)} = f_{Lj}(r, t) \quad (179)$$

$$[J_{\theta j(0,1,0)}^{(\alpha 2\gamma)} + \frac{h_2}{2} J_{\theta j(0,0,0)}^{(\alpha 2\gamma)}]^{(p,N_q,s)} = f_{Rj}(r, t) \quad (180)$$

These relations readily imply the following equations:

$$[\rho^{(\alpha 1\gamma)} \frac{h_1^2}{12} \ddot{W}_{1(010)}^{(\alpha 1\gamma)} - \rho^{(\alpha 1\gamma)} \frac{h_1^3}{40} \ddot{W}_{1(020)}^{(\alpha 1\gamma)}]^{(p,1,s)} = [-j_{\theta r(0,1,0)}^{(\alpha 1\gamma)} + \frac{h_1}{2} j_{\theta r(0,0,0)}^{(\alpha 1\gamma)}]^{(p,1,s)} + f_{Lr}(r, t) \quad (181)$$

$$[\rho^{(\alpha 1 \gamma)} \frac{h_1^2}{12} \ddot{W}_{2(010)}^{(\alpha 1 \gamma)} - \rho^{(\alpha 1 \gamma)} \frac{h_1^3}{40} \ddot{W}_{2(020)}^{(\alpha 1 \gamma)}]^{(p,1,s)} = [-j_{\theta\theta(0,1,0)}^{(\alpha 1 \gamma)} + \frac{h_1}{2} j_{\theta\theta(0,0,0)}^{(\alpha 1 \gamma)}]^{(p,1,s)} + f_{L\theta}(r, t) \quad (182)$$

$$[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{12} \ddot{W}_{1(010)}^{(\alpha 2 \gamma)} + \rho^{(\alpha 2 \gamma)} \frac{h_2^3}{40} \ddot{W}_{1(020)}^{(\alpha 2 \gamma)}]^{(p,N_q,s)} = [-j_{\theta r(0,1,0)}^{(\alpha 2 \gamma)} - \frac{h_2}{2} j_{\theta r(0,0,0)}^{(\alpha 2 \gamma)}]^{(p,N_q,s)} + f_{Rr}(r, t) \quad (183)$$

$$[\rho^{(\alpha 2 \gamma)} \frac{h_2^2}{12} \ddot{W}_{2(010)}^{(\alpha 2 \gamma)} + \rho^{(\alpha 2 \gamma)} \frac{h_2^3}{40} \ddot{W}_{2(020)}^{(\alpha 2 \gamma)}]^{(p,N_q,s)} = [-j_{\theta\theta(0,1,0)}^{(\alpha 2 \gamma)} - \frac{h_2}{2} j_{\theta\theta(0,0,0)}^{(\alpha 2 \gamma)}]^{(p,N_q,s)} + f_{R\theta}(r, t) \quad (184)$$

If, on the other hand, time-dependent displacement is imposed at the boundary, the same averaging procedure is employed to establish the required expressions. For example, if the radial displacement  $u_r$  is imposed at the right boundary  $\theta = \Theta$ , the following expression is obtained:

$$[\ddot{W}_{1(000)}^{(\alpha 2 \gamma)} + \frac{h_2}{2} \ddot{W}_{1(010)}^{(\alpha 2 \gamma)} + \frac{h_2^2}{4} \ddot{W}_{1(020)}^{(\alpha 2 \gamma)}]^{(p,N_q,s)} = \ddot{f}_{Rr}(r, t) \quad (185)$$

**4.15. Imposition of Plane Strain or Generalized Plane Strain Conditions .** So far, we have established 104 relations in every cell for the determination of the 112 unknowns  $\ddot{W}_{j(lmn)}^{(\alpha\beta\gamma)}$  in this cell. The final set of 8 relations is determined from the imposition of either a plane strain or generalized plane strain condition in the periodic direction.

By applying a homogenization procedure in the periodic z-direction, it can be shown that the following relation holds

$$\bar{\epsilon}_{zz} = \frac{\partial W_{3(000)}^{(\alpha\beta\gamma)}}{\partial z} \quad (186)$$

for all  $\alpha, \beta, \gamma = 1, 2$  in all the cells, where  $\bar{\epsilon}_{zz}$  is the far field average normal strain in the z-direction. Utilizing this relation, we can reduce the number of unknowns in each cell from 112 to 104 by replacing the 8 unknowns  $W_{3(000)}^{(\alpha\beta\gamma)}$  with the single new unknown  $\bar{\epsilon}_{zz}$ . Consequently, the number of unknowns in each cell is 104, and the number of unknowns in the entire FGM composite comprised of  $N_p N_q$  cells is also  $104 N_p N_q + 1$ .

The global axial strain is related to the local strain  $\epsilon_{zz}^{(\alpha\beta\gamma)} = W_{3(001)}^{(\alpha\beta\gamma)}$  as follows

$$\bar{\epsilon}_{zz} = \frac{1}{V} \sum_{p=1}^{N_p} \sum_{q=1}^{N_q} \sum_{\alpha, \beta, \gamma=1}^2 d_{\alpha}^{(p)} h_{\beta}^{(q)} l_{\gamma} \epsilon_{zz}^{(\alpha\beta\gamma)} \quad (187)$$

where  $V = D(R_0 + R_1)\Theta(l_1 + l_2)/2$  is the total volume of the composite.

Under plane strain condition this far field axial strain vanishes, namely,

$$\bar{\epsilon}_{zz} = 0 \quad (188)$$

For a generalized plane strain situation, on the other hand, the average normal stress in the z-direction vanishes:

$$\bar{\sigma}_{zz} = \frac{1}{V} \sum_{p=1}^{N_p} \sum_{q=1}^{N_q} \sum_{\alpha, \beta, \gamma=1}^2 d_{\alpha}^{(p)} h_{\beta}^{(q)} l_{\gamma} S_{zz(0,0,0)}^{(\alpha\beta\gamma)} = 0 \quad (189)$$

and the 2nd time derivative of this equation forms the required additional relation.

**4.16. Summary of the Governing Equations.** In summary, we have altogether  $104N_pN_q + 1$  equations for the determination of the 2nd time derivatives of the  $104N_pN_q + 1$  displacement coefficients,  $\ddot{W}_{j(lmn)}^{(\alpha\beta\gamma)}$ , and the 2nd time derivative of the unknown global axial strain  $\ddot{\epsilon}_{zz}$ . This system of equations can be represented in the following compact form

$$\mathbf{A}\ddot{\mathbf{U}}(t) = \mathbf{f}(t) + \mathbf{g}(t) \quad (190)$$

where the structural stiffness matrix  $\mathbf{A}$  contains information on the geometry and thermomechanical properties of the individual subcells  $(\alpha\beta\gamma)$  within the cells comprising the functionally graded material. The displacement coefficient vector  $\ddot{\mathbf{U}}(t)$  contains the  $104N_pN_q + 1$  2nd time derivatives of the unknowns:

$$\ddot{\mathbf{U}}(t) = [\ddot{\mathbf{U}}_{11}^{(\alpha\beta\gamma)}(t), \dots, \ddot{\mathbf{U}}_{N_pN_q}^{(\alpha\beta\gamma)}(t), \ddot{\epsilon}_{zz}(t)] \quad (191)$$

where

$$\ddot{\mathbf{U}}_{pq}^{(\alpha\beta\gamma)}(t) = [\ddot{W}_{1(lmn)}^{(\alpha\beta\gamma)}(t), \ddot{W}_{2(lmn)}^{(\alpha\beta\gamma)}(t), \ddot{W}_{3(lmn)}^{(\alpha\beta\gamma)}(t)]_{pq} \quad l, m, n = 0, 1, 2. \quad (192)$$

The mechanical force vector  $\mathbf{f}(t)$  contains information on the mechanical boundary conditions and the thermal loading effects generated by the applied temperature. In addition, the inelastic force vector  $\mathbf{g}(t)$  appearing on the right-hand side of eqn (190) contains inelastic effects given in terms of the integrals of inelastic strain distributions that are represented by  $R_{ij}^{(\alpha\beta\gamma)}(l, m, n)$ .

The field variables that are expressed by the vector  $\ddot{\mathbf{U}}(t)$  can be determined by integrating the above equation explicitly in a step-by-step timewise manner. To this end, let us introduce a time increment  $\Delta t$ . The time integration at time  $t$  yields

$$\mathbf{U}(t + \Delta t) = (\Delta t)^2 \mathbf{A}^{-1}[\mathbf{f}(t) + \mathbf{g}(t)] + 2\mathbf{U}(t) - \mathbf{U}(t - \Delta t) \quad (193)$$

This difference expression which approximates the above 2nd order differential equation provides the displacement coefficients at the next time step  $t + \Delta t$  from the already known quantities at the current time  $t$  and the previous time step  $t - \Delta t$  (assuming that at time  $t = 0$  the thermoelastic field distribution in the composite is known). Note that for time-independent material properties the matrix  $\mathbf{A}$  has to be inverted just one time. This procedure is continued until the desired final time is reached.

## 5. APPLICATION

To demonstrate the potential of the outlined theoretical framework of the two-dimensional higher-order theory for cylindrical functionally graded materials with dynamic loading capability, a small computer code was developed to simulate one-dimensional wave propagation due to impulse loading in a layered half-plane as a special case. The half-plane consists of alternating layers of steel and polymeric material (PMMA) 0.025 and 0.0784 cm thick, respectively. Both materials are assumed to be linearly elastic with the relevant material parameters given in Table 1.

	Young's modulus, E (dyne/cm <sup>2</sup> )	Mass density, $\rho$ (gm/cm <sup>3</sup> )
Stainless steel	$1.258 \times 10^{12}$	7.9
PMMA	$0.089 \times 10^{12}$	1.15

Table 1. Material constants of a half-plane consisting of alternating steel and PMMA layers.

For an impacted linearly elastic composite by a spatially uniform loading applied normal to the layering, an exact solution by ray theory can be constructed, Ref. [23]. Figures 2 and 3 illustrate the predictions by the ray theory taken from this reference for the case when the half-plane is impacted by a unit impulsive normal stress applied at the center of the first layer (steel) at time  $t = 0$  sec. The normal stress response with time is detected at the centers of the 11th, Fig. 2, and 12th layers, Fig. 3. The corresponding results generated by the computer code developed for this particular one-dimensional wave propagation case are included in the bottom portions of these figures. Excellent agreement between the exact ray-theory solution and the higher-order theory predictions is observed.

## 6. PLANS FOR FUTURE WORK

The completed generalization of the higher-order theory for cylindrical functionally graded materials with dynamic impact loading capability completely fulfills the objectives of the Grant NAG3-2411. As demonstrated in the above section, full benefit of the developed theoretical framework will be realized upon development of a general computer code enabling simulation of two-dimensional wave propagation in bi-directionally graded cylindrical structural components in the  $r - \theta$  plane.

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## 8. APPENDIX

The nonzero strain coefficients  $e_{ij}^{(\alpha\beta\gamma)}$  in the Legendre polynomial expansion of the strain field in subcell  $(\alpha\beta\gamma)$  of cell  $(p, q, s)$  are given in terms of the displacement field microvariables by

$$\begin{aligned}
 e_{rr(0,0,0)}^{(\alpha\beta\gamma)} &= W_{1(100)}^{(\alpha\beta\gamma)} \\
 e_{rr(1,0,0)}^{(\alpha\beta\gamma)} &= \frac{\sqrt{3}d_\alpha}{2} W_{1(200)}^{(\alpha\beta\gamma)} \\
 e_{\theta\theta(0,0,0)}^{(\alpha\beta\gamma)} &= W_{2(010)}^{(\alpha\beta\gamma)} + \frac{1}{R(\alpha\beta\gamma)} W_{1(000)}^{(\alpha\beta\gamma)} \\
 e_{\theta\theta(1,0,0)}^{(\alpha\beta\gamma)} &= \frac{d_\alpha}{2\sqrt{3}R(\alpha\beta\gamma)} W_{1(100)}^{(\alpha\beta\gamma)} \\
 e_{\theta\theta(0,1,0)}^{(\alpha\beta\gamma)} &= \frac{h_\beta}{2\sqrt{3}R(\alpha\beta\gamma)} W_{1(010)}^{(\alpha\beta\gamma)} + \frac{\sqrt{3}h_\beta}{2} W_{2(020)}^{(\alpha\beta\gamma)} \\
 e_{\theta\theta(2,0,0)}^{(\alpha\beta\gamma)} &= \frac{d_\alpha^2}{4\sqrt{5}R(\alpha\beta\gamma)} W_{1(200)}^{(\alpha\beta\gamma)} \\
 e_{\theta\theta(0,2,0)}^{(\alpha\beta\gamma)} &= \frac{h_\beta^2}{4\sqrt{5}R(\alpha\beta\gamma)} W_{1(020)}^{(\alpha\beta\gamma)} \\
 e_{\theta\theta(0,0,2)}^{(\alpha\beta\gamma)} &= \frac{l_\gamma^2}{4\sqrt{5}R(\alpha\beta\gamma)} W_{1(002)}^{(\alpha\beta\gamma)} \\
 e_{zz(0,0,0)}^{(\alpha\beta\gamma)} &= W_{3(001)}^{(\alpha\beta\gamma)} \\
 e_{r\theta(0,0,0)}^{(\alpha\beta\gamma)} &= \frac{1}{2} [W_{2(100)}^{(\alpha\beta\gamma)} + W_{1(010)}^{(\alpha\beta\gamma)}] - \frac{1}{2R(\alpha\beta\gamma)} W_{2(000)}^{(\alpha\beta\gamma)} \\
 e_{r\theta(1,0,0)}^{(\alpha\beta\gamma)} &= \frac{\sqrt{3}d_\alpha}{4} W_{2(200)}^{(\alpha\beta\gamma)} - \frac{d_\alpha}{4\sqrt{3}R(\alpha\beta\gamma)} W_{2(100)}^{(\alpha\beta\gamma)} \\
 e_{r\theta(0,1,0)}^{(\alpha\beta\gamma)} &= \frac{\sqrt{3}h_\beta}{4} W_{1(020)}^{(\alpha\beta\gamma)} - \frac{h_\beta}{4\sqrt{3}R(\alpha\beta\gamma)} W_{2(010)}^{(\alpha\beta\gamma)} \\
 e_{r\theta(2,0,0)}^{(\alpha\beta\gamma)} &= -\frac{d_\alpha^2}{8\sqrt{5}R(\alpha\beta\gamma)} W_{2(200)}^{(\alpha\beta\gamma)} \\
 e_{r\theta(0,2,0)}^{(\alpha\beta\gamma)} &= -\frac{h_\beta^2}{8\sqrt{5}R(\alpha\beta\gamma)} W_{2(020)}^{(\alpha\beta\gamma)} \\
 e_{r\theta(0,0,2)}^{(\alpha\beta\gamma)} &= -\frac{l_\gamma^2}{8\sqrt{5}R(\alpha\beta\gamma)} W_{2(002)}^{(\alpha\beta\gamma)} \\
 e_{rz(0,0,1)}^{(\alpha\beta\gamma)} &= \frac{\sqrt{3}l_\gamma}{4} W_{1(002)}^{(\alpha\beta\gamma)} \\
 e_{\theta z(0,0,1)}^{(\alpha\beta\gamma)} &= \frac{\sqrt{3}l_\gamma}{4} W_{2(002)}^{(\alpha\beta\gamma)}
 \end{aligned}$$

The nonzero thermal stress coefficients  $\tau_{ij(l,m,n)}^{T(\alpha\beta\gamma)}$  in the Legendre polynomials expansion of the thermal stress in subcell  $(\alpha\beta\gamma)$  of cell  $(p, q, s)$  are given in terms of of the temperature field microvariables  $T_{(l,m,n)}^{(\alpha\beta\gamma)}$  by

$$\tau_{rr(0,0,0)}^{T(\alpha\beta\gamma)} = \Gamma_{rr}^{(\alpha\beta\gamma)} T_{(000)}^{(\alpha\beta\gamma)}$$

$$\tau_{rr(1,0,0)}^{T(\alpha\beta\gamma)} = \Gamma_{rr}^{(\alpha\beta\gamma)} \frac{d_\alpha}{2\sqrt{3}} T_{(100)}^{(\alpha\beta\gamma)}$$

$$\tau_{rr(0,1,0)}^{T(\alpha\beta\gamma)} = \Gamma_{rr}^{(\alpha\beta\gamma)} \frac{h_\beta}{2\sqrt{3}} T_{(010)}^{(\alpha\beta\gamma)}$$

$$\tau_{rr(2,0,0)}^{T(\alpha\beta\gamma)} = \Gamma_{rr}^{(\alpha\beta\gamma)} \frac{d_\alpha^2}{4\sqrt{5}} T_{(200)}^{(\alpha\beta\gamma)}$$

$$\tau_{rr(0,2,0)}^{T(\alpha\beta\gamma)} = \Gamma_{rr}^{(\alpha\beta\gamma)} \frac{h_\beta^2}{4\sqrt{5}} T_{(020)}^{(\alpha\beta\gamma)}$$

$$\tau_{rr(0,0,2)}^{T(\alpha\beta\gamma)} = \Gamma_{rr}^{(\alpha\beta\gamma)} \frac{l_\gamma^2}{4\sqrt{5}} T_{(002)}^{(\alpha\beta\gamma)}$$

with similar expressions for  $\tau_{\theta\theta(l,m,n)}^{T(\alpha\beta\gamma)}$  and  $\tau_{zz(l,m,n)}^{T(\alpha\beta\gamma)}$ .

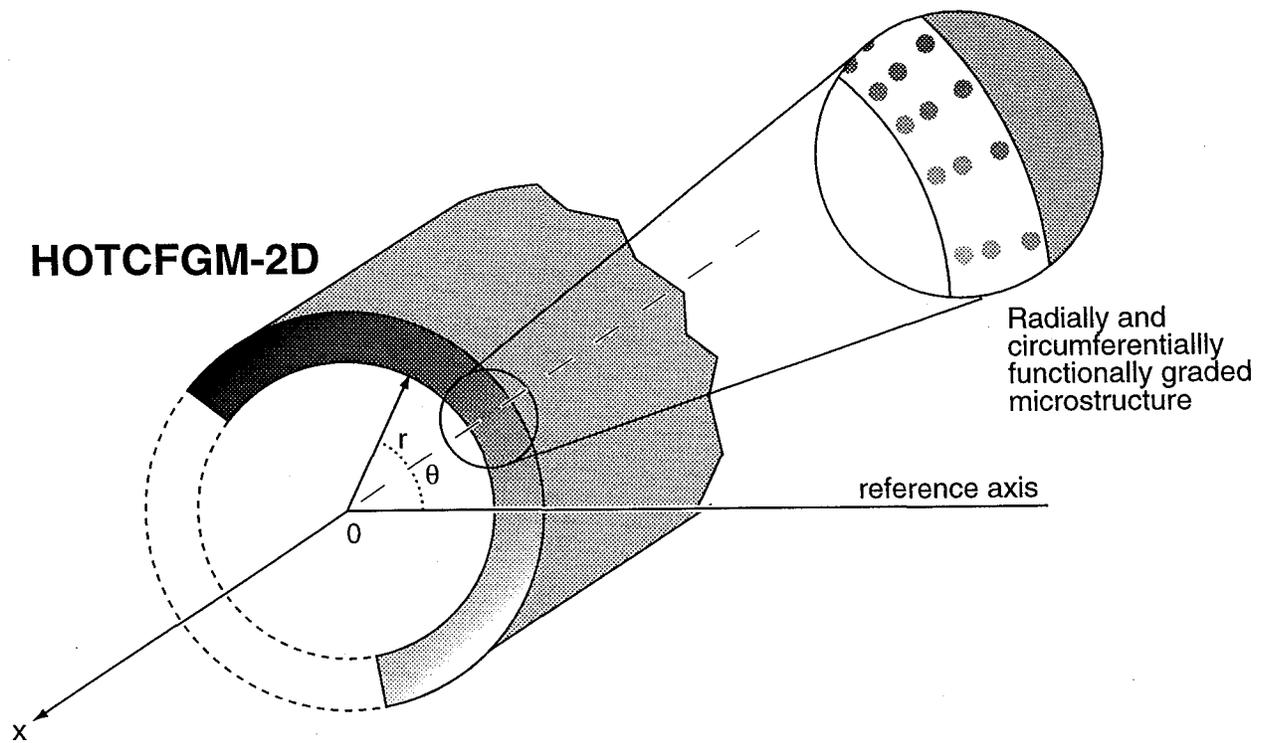


Figure 1. A geometric model for the higher-order theory for cylindrical functionally graded materials/structures (HOTCFGM-2D).

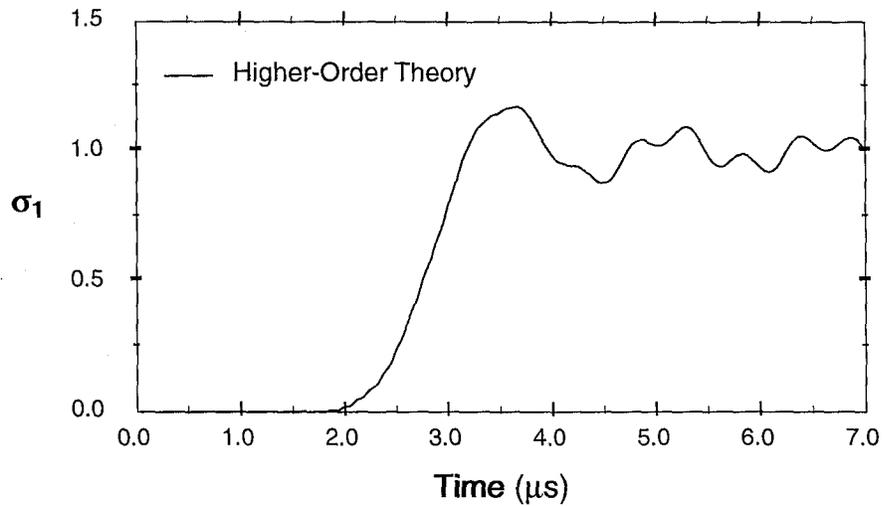
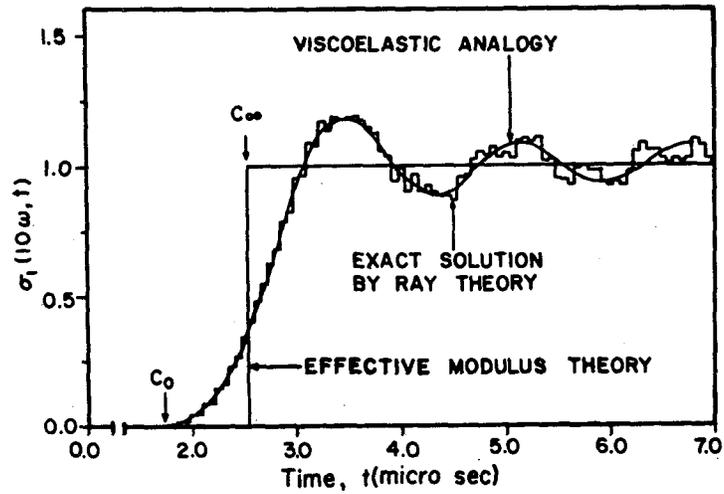


Figure 2. Stress at the center of the 11th layer due to a unit step stress applied at the center of the first (top) layer: exact ray theory predictions from Ref. [23] (top figure); higher-order theory predictions (bottom figure). The subscript 1 in  $\sigma_1$  indicates steel layer (layer 1) in the repeating unit cell sequence of the layered half-plane.

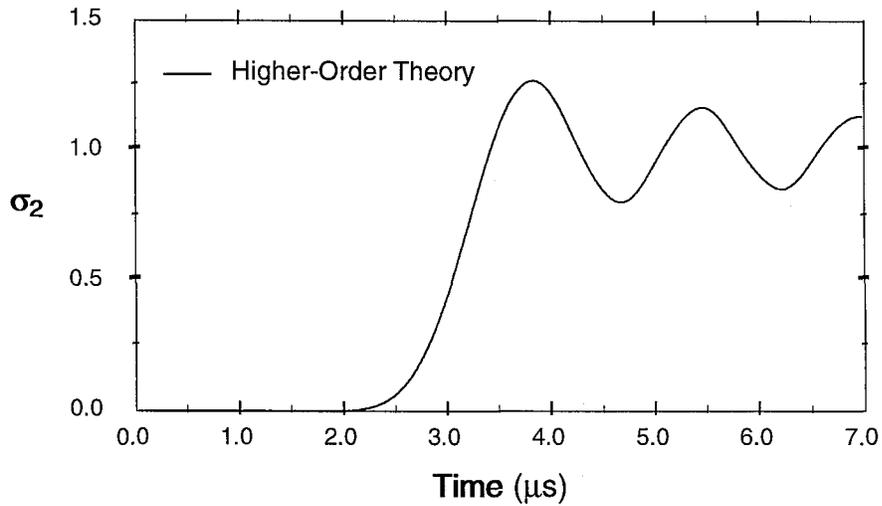
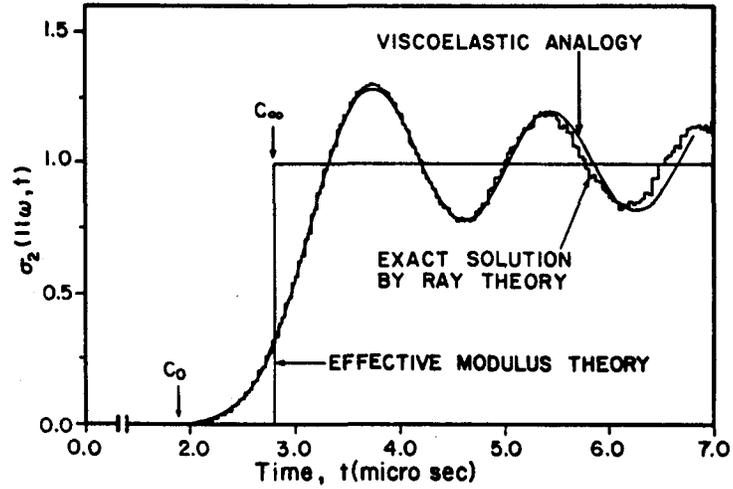


Figure 3. Stress at the center of the 12th layer due to a unit step stress applied at the center of the first (top) layer: exact ray theory predictions from Ref. [23] (top figure); higher-order theory predictions (bottom figure). The subscript 1 in  $\sigma_2$  indicates PMMA layer (layer 2) in the repeating unit cell sequence of the layered half-plane.

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