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# Chaos in a Fractional Order Chua System

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## Summary

This report studies the effects of fractional dynamics in chaotic systems. In particular, Chua's system is modified to include fractional order elements. Varying the total system order incrementally from 2.6 to 3.7 demonstrates that systems of "order" less than three can exhibit chaos as well as other nonlinear behavior. This effectively forces a clarification of the definition of order which can no longer be considered only by the total number of differentiations or by the highest power of the Laplace variable.

## Introduction

It is well known that chaos cannot occur in continuous-time systems of order less than three. This assertion is based on the usual concepts of order, such as the number of states in a system, the highest power of the Laplace variable,  $s$ , in the system, or the total number of separate differentiations or integrations in a system. Unfortunately, these concepts of order do not directly relate to systems having fractional order components. The purpose of this report is to demonstrate that systems of order less than three, as defined in the usual way, can still display chaotic behavior. The next section provides a brief review of fractional calculus. Useful approximations for these fractional operators follow. Finally, an example is given which demonstrates that systems of order less than three can display chaos. This is both shown experimentally via simulations and predicted analytically using the describing function method.

## Review of Fractional Operators

The idea of fractional integrals and derivatives has been known since the development of the regular calculus, with the

first reference probably being associated with Leibniz in 1695 (Oldham and Spanier, 1974, page 3). Although not well known to most engineers, the fractional calculus has been considered by prominent mathematicians (Courant and Hilbert, 1953) as well as the "engineers" of the operational calculus (Heaviside, 1971; and Bush, 1929). In fact several textbooks written before 1960 have some small section on fractional calculus (Goldman, 1949; Holbrook, 1966; Starkey, 1954; Carslaw and Jeager, 1948; Scott, 1955; and Mikusinski, 1959). An outstanding historical survey can be found in Oldham and Spanier (1974) who also give what is unquestionably the most readable and complete mathematical presentation of the fractional calculus. Other bound discussions of the area are given by Ross (1975), McBride (1979), and McBride and Roach (1985). Unfortunately, many of the results in the fractional calculus are given in the language of advanced analysis and are not readily accessible to the general engineering community.

Many systems are known to display fractional order dynamics. Probably the first physical system to be widely recognized as one demonstrating fractional behavior is the semi-infinite lossy (RC) line. The current into the line is equal to the half-derivative of the applied voltage; that is, the impedance is

$$V(s) = \frac{1}{\sqrt{s}} I(s)$$

Although this system was studied by many, Heaviside (1971) considered it extensively using the operational calculus. He states "there is a universe of mathematics lying in between the complete differentiations and integrations" and that "fractional (operators) push themselves forward sometimes, and are just as real as the others." Another equivalent system is the diffusion of heat into a semi-infinite solid. Here the temperature looking in from the boundary is equal to the half integral of the heat rate there. Other systems that are known to display fractional order

dynamics are viscoelastic systems (Bagley and Calico, 1991; Koeller, 1984; Koeller, 1986; Skaar, Michel, and Miller, 1988; Lopez-Marcos, 1990); colored noise (Mandelbrot, 1967); electrode-electrolyte polarization (Ichise, Nagayanagi, and Kojima, 1971; Sun, Onaral, and Tsao, 1984); dielectric polarization (Sun, Abdelwahab, and Onaral, 1984); boundary layer effects in ducts (Sugimoto, 1991); and electromagnetic waves (Heaviside, 1971). Because many of these systems depend upon specific material and chemical properties, it is expected that a wide range of fractional order behaviors are possible using different materials.

Two commonly used definitions for the general fractional differintegral are the Grunwald definition and the Riemann-Liouville definition (Oldham and Spanier, 1974). The Riemann-Liouville definition of the fractional integral is given here as

$$\frac{d^q f}{dt^q} = \frac{1}{\Gamma(-q)} \int_0^t \frac{f(\tau)}{(t-\tau)^{q+1}} d\tau, \quad q < 0$$

where  $q$  can have noninteger values, and thus the name fractional differintegral. Notice that the definition is based on integration and more importantly is a convolution integral for  $q < 0$ . When  $q > 0$ , then the usual integer  $n$ th derivative must be taken of the fractional  $(q - n)$ th integral, and yields the fractional derivative of order  $q$  as

$$\frac{d^q f}{dt^q} = \frac{d^n}{dt^n} \left[ \frac{d^{q-n} f}{dt^{q-n}} \right], \quad q > 0 \text{ and } n \text{ an integer } > q$$

This appears so vastly different from the usual intuitive definition of derivative and integral that the reader must abandon the familiar concepts of slope and area and attempt to get some new insight (which still remains elusive). This is discussed further in Lorenzo, C.F.; and Hartley, T.T.: On Conceptualization, Initialization, and Applications in Fractional Calculus (to be published).

Fortunately, the basic engineering tool for analyzing linear systems, the Laplace transform, is still applicable and works as one would expect; that is,

$$L \left\{ \frac{d^q f(t)}{dt^q} \right\} = s^q L \{ f(t) \} - \sum_{k=0}^{n-1} s^k \left[ \frac{d^{q-1-k} f(t)}{dt^{q-1-k}} \right]_{t=0}, \text{ for all } q$$

where  $n$  is an integer such that  $n - 1 < q < n$  (Oldham and Spanier, 1974). If the initial conditions are considered to be zero, this formula reduces to the more expected and comforting form

$$L \left\{ \frac{d^q f(t)}{dt^q} \right\} = s^q L \{ f(t) \}$$

Amazingly enough, one of the most difficult obstacles in the practical application of the fractional calculus is the initial condition problem. As long as a given system is at rest, at the zero equilibrium, at time zero, the fractional initial value problem is readily solved using standard Laplace transform methods (all initial condition terms are zero). Unfortunately, the fractional derivative operator starts rather abruptly at time zero, so that any nonzero initial value for a function will appear as a discontinuity and translate directly into a  $t^{-r}$  term, which has an annoying singularity at time zero using the appropriate power,  $r$ . This is not necessarily a problem, unless the desired initial value is not infinity. Bagley (1988) addresses this problem by creating a modified fractional derivative operator that essentially subtracts out the singularity. The problem is further studied by Hartley, T.T.; and Lorenzo, C.F.: Insights Into the Fractional Initiative Value Problems (to be published) by relating it back to the semi-infinite line problem.

Bagley (1988) has also extended the initial value problem to fractional state space systems. Here the idea of state no longer gives all past and future knowledge of the system behavior via some stored pseudo-energy. In fact, the number of these fractional states is somewhat arbitrary and dependent only upon what the user has chosen as the base fractional derivative.

Understanding the possible dynamic behavior of linear fractional order systems is fundamental to the development of future applications. Progress in this area has been fairly slow, however, since there was no known general fractional order impulse response with which to perform convolution. Recently, Bagley (1988) has shown that the impulse responses of fractional order systems are related to the Mittag-Leffler function (Erdelyi, et al. 1955), which is effectively the fractional order analog of the exponential function. With this knowledge, it has been possible to better clarify the time responses associated with fractional order systems. Impulse responses, step responses, and initial condition responses for some general fractional order systems can be found in Hartley, T.T.; and Lorenzo, C.F.: The Solution to a General Linear Fractional Order Initial Value Problem (to be published).

## The Concept of System Order

As the concept of "order" is central to the understanding of fractional systems, some discussion of this concept now follows. In this discussion, it will be assumed that the systems being considered are single-input-single-output, that their representations are minimal in the usual sense (Kailath, 1980), and that they are linear.

Mathematical order is defined as the highest derivative occurring in a given differential equation. The concept of mathematical order is applicable to both ordinary and fractional differential equations. Normally, when the word “order” is used without a qualifier, it implies the meaning of mathematical order.

For linear dynamic systems that are described by ordinary differential equations (i.e., of integer mathematical order), the system mathematical order implies, or is equivalent to, the following:

- (1) The highest derivative in the ordinary differential equation
- (2) The highest power of the Laplace variable,  $s$ , in the characteristic equation
- (3) The number of initializing constants required for the differential equation
- (4) The number of singularities in the characteristic equation
- (5) The length of the state vector
- (6) The number of energy storage elements
- (7) The number of independent spatial directions in which a trajectory can move
- (8) The number of devices that add  $90^\circ$  sinusoidal steady state phase lag
- (9) The number of devices that retain some memory of the past

The utility of the definition of mathematical order is that it infers all the system characteristics for systems with only integer order components.

Thus the benefit of having a definition for order for linear ordinary differential equations is that it allows a direct understanding of the behavior of a given dynamic system. Unfortunately, for fractional differential equations, the order of the highest derivative does not infer (or is not equal to) all of the previously mentioned properties. Indeed, the most important characteristic of order in integer order ordinary differential equations is probably item (3) in the previous list (i.e., it dictates the number of initializing constants which together with the differential equations allow prediction of the future behavior). In systems terminology, this information provides the initial “state” of the system being analyzed. Clearly, the order of the highest derivative in a fractional differential equation does not have this property, nor does it predict the associated number of energy/memory elements associated with the fractional differential equation, nor does it infer the number of integrations (even fractional) required to solve or simulate the given fractional differential equation. Thus the issue of order and the information required together with the fractional differential equation to predict future behavior is fundamental and is expected to be treated in detail at a later time.

## Approximation of Fractional Operators

The standard definitions of the fractional differintegral do not allow direct implementation of the operator in time-domain simulations of complicated systems with fractional elements. Thus, in order to effectively analyze such systems, it is necessary to develop approximations to the fractional operators using the standard integer order operators. In the work that follows, the approximations are effected in the Laplace  $s$ -variable. The resulting approximations provide sufficient accuracy for time domain hardware implementations.

Some work has been done in this area already, but it has not been highly organized. Oldham and Spanier (1974) and Piche (1992) give several discrete-time approximations based on numerical quadrature. In continuous time, engineers have used network theory approximations (Carlson and Halijak, 1964; Steiglitz, 1964; Carlson and Halijak, 1961; and Halijak, 1964). More recently Oldham and Spanier (1974), Ichise, Nagayanagi, and Kojima, 1971; and Charef, et al. (1992) have developed other network theory approximations. Even more recently, a discrete-time fractional calculus has been developed similar to the theory of linear multistep methods for numerical integration (Lubich, 1985, 1986, 1988a, and 1988b).

The approximation approach taken here is that of Charef, et al. (1992). Basically the idea is to approximate the system behavior in the frequency domain. This is done for a given  $q$  by creating an approximation with Bode magnitude response roll off of 20 times  $q$  db/dec, which will consequently have a phase shift of approximately 90 times  $q$  degrees over the required frequency band. This approximation is created by choosing an initial breakpoint (the low frequency accuracy limit of the approximation), the allowable error in db's, and the number of  $s$ -plane poles in the approximation. The high frequency limit of the usable bandwidth can be varied by changing the allowable error and the number of poles. Thus an approximation of any desired accuracy over any frequency band can be achieved. Table I gives approximations for  $1/s^q$  with  $q = 0.1$  to  $0.9$  in steps of 0.1. These were obtained by trial and error and are reasonably good from 0.01 to 100 rad/sec. These approximations are used in the study that follows.

## A Fractional Chua System

Chua's system is well known and has been extensively studied. The particular form to be considered here was presented by Hartley (1989) and used further for the study of Hartley and Mossayebi (1993). This system is different from the usual Chua system in that the piecewise-linear nonlinearity is replaced by an appropriate cubic nonlinearity which yields very similar behavior. It is represented in state space form as

$$\dot{x} = \alpha \left[ y + \frac{x - 2x^3}{7} \right]$$

$$\dot{y} = x - y + z$$

$$\dot{z} = -\frac{100y}{7} = -\beta y$$

It is studied here in two different system representations as discussed in the following sections. In all cases studied,  $\beta$  is defined to be  $100/7$  and  $\alpha$  is allowed to vary.

### State Space Configuration

To study the effect of fractional derivatives on the dynamics of this system, the state space configuration (fig. 1(a)) was considered first. Here, the vector derivative was replaced by a vector fractional derivative as follows:

$$\frac{d^q x}{dt^q} = \alpha \left[ y + \frac{x - 2x^3}{7} \right]$$

$$\frac{d^q y}{dt^q} = x - y + z$$

$$\frac{d^q z}{dt^q} = -\frac{100y}{7} = -\beta y$$

Simulations were then performed using  $q = 0.8, 0.9, 1.0,$  and  $1.1$ . The approximations from table I were used for the simulations of the appropriate  $q$ th integrals. When  $q < 1$ , then the approximations were used directly. It should further be noted that approximations used in the simulations for  $1/s^q$ , when  $q > 1$ , are obtained by using  $1/s$  times the approximation for  $1/s^{q-1}$  from table I.

The results from this state space study verified that chaos could indeed occur in a system of mathematical order less than 3. This was determined by computing the Lyapunov exponents for each of the simulations with  $q = 0.9, 1.0,$  and  $1.1$ , using the method of Benettin, et al. (1990). Chaos is indicated when any of the Lyapunov exponents is greater than zero. These results are given in table II where the largest Lyapunov exponents are given as a function of system order. In each case, the second exponent was near zero. The 2.7 order system approximation had an additional six negative exponents which were not listed. Also the 3.3 order system approximation was so large that it

prohibited a timely calculation of any exponents but the first. Since the order of this system was greater than three, these calculations were not pursued. In all cases, the one positive exponent clearly indicated that the system was behaving chaotically. The numerical simulations also indicated that the lower limit of the vector fractional derivative  $q$  was between 0.8 and 0.9 for this system to remain capable of generating chaos. The lowest value obtained for mathematical order to yield chaos was 2.7 using the  $q = 0.9$  fractional vector derivative. No upper limit was obtained. Phase plane plots for these systems are given in figure 2.

### Feedback Configuration

The feedback configuration is now considered. To change the total system mathematical order, the separated  $1/s$  in figure 1(b) was allowed to change powers, that is,

$$\frac{1}{s} \rightarrow \frac{1}{s^q}$$

A variety of simulations were performed on the resulting systems as discussed subsequently. Here, the approximations from table I were used to represent the fractional integral where again the approximations for  $1/s^q$ , when  $q > 1$ , were obtained by using  $1/s$  times the approximation for  $1/s^{q-1}$ .

Bifurcation diagrams for several of these systems are given in figure 3. Here, a particular value of  $q$  was chosen, and the parameter was varied to obtain the particular bifurcation plot. These diagrams were generated by simulation using Euler's method and a simulation timestep of 0.001. These were verified by further reducing the timestep by an order of magnitude with little change in the overall bifurcation structure. To obtain these diagrams, the values of the output  $x$ -variable were plotted whenever its slope changed sign. Although it is believed that the bifurcation diagrams are reasonably accurate and are sufficiently accurate for this particular study, more correct diagrams could possibly be obtained by using more accurate approximations of the fractional derivative than those given in table I or a more accurate simulation. Observation of the bifurcation diagrams indicates behavior similar to that from the state space study. For the feedback configuration, decreasing the power of  $s$  shifts the bifurcation diagram to the right as a function of  $\alpha$ , while the converse is also true. The limits on the system mathematical order to have a chaotic response as measured from the bifurcation diagrams are approximately  $2.5 < n < 3.8$ . The overall behavior from the simulation studies is summarized in figure 4.

An advantage to the feedback configuration is that it allows easy system analysis using describing functions, as discussed in Hartley and Mossayebi (1993). Here the idea is that the

frequency response of the linear block in the feedback configuration is plotted in the Nyquist plane, together with minus one over the appropriate describing function of the nonlinearity, as in figure 5. The fractional order integral in the loop is handled directly by taking the frequency response on the primary Riemann sheet and essentially poses no complication or confusion in application of the describing function approach. In other words, the fact that fractional powers of  $s$  are present does not require any frequency domain approximation as in the time-domain simulation; rather the fractional powers of  $s$  can be used as is in computing the frequency response of the linear block. In Hartley and Mossayebi (1993), it is shown that the important points from the nonlinearity of this system in the Nyquist plane are

- (1)  $\Re[H(j\omega)] > -3.5$ ,  $\Im[H(j\omega)] = 0$ , which indicates two stable points at  $x = \pm\sqrt{0.5}$ .
- (2)  $\Re[H(j\omega)] \leq -3.5$ ,  $\Im[H(j\omega)] = 0$ , which indicates a Hopf bifurcation of the stable points of item (1) into a limit cycle.
- (3)  $\Re[H(j\omega)] \leq -7$ ,  $\Im[H(j\omega)] = 0$ , which indicates that period doubling of the limit cycle of item (2) occurs (this progresses into spiral chaos).
- (4)  $\Re[H(j\omega)] \leq -14$ ,  $\Im[H(j\omega)] = 0$ , which indicates merging of the spiral chaos into the double scroll behavior.

Extinction of the double scroll (meaning its disappearance) is not directly predicted using the describing function approach, but a reasonable approximate value is  $\Re[H(j\omega)] \leq -23$ ,  $\Im[H(j\omega)] = 0$ . A diagram indicating the usage of the describing function is given in figure 5.

Using these results and varying the power of the integrator in the loop allowed a theoretical prediction of the simulation results of figure 4. These theoretical results are given in figure 6. It should be noted that the qualitative features are very well predicted using the describing function approach, and that the quantitative results are reasonably close. Furthermore, for mathematical system order less than approximately 2.85, the describing function approach predicted the appearance of a stable and unstable limit cycle as  $\alpha$  increased (via an apparent saddle node bifurcation). These limit cycles coexist with each of the stable fixed points. Eventually, as  $\alpha$  increased further, the unstable cycles merged with the stable fixed points via a subcritical Hopf bifurcation, leaving unstable fixed points. This entire process basically became a supercritical Hopf bifurcation for mathematical order greater than 2.85. This was then verified in the simulations with this bifurcation structure occurring for mathematical system order less than approximately 2.75. In fact, for the mathematical order equal to 2.6, the simulation showed the points at  $x = \pm\sqrt{0.5}$  to be stable and each coexisting with spiral chaos. It is a true testament to the utility of the describing function approach that it could predict the behavior of this system as accurately as it does.

## Concluding Remarks

This report has introduced the idea of fractional derivatives from the dynamic systems viewpoint. It has been demonstrated that the usual idea of system order must be modified when fractional derivatives are present. The usual approach of calculating the mathematical system order by determining the highest derivative in the system does not work in this situation.

It has been further demonstrated that chaos, as well as the other usual nonlinear dynamic phenomena, can occur in systems with mathematical order less than three via Chua's system. This is surprising given the usual nonlinear system paradigms concerning chaos and order. It is not clear at this point whether the chaos in fractional order systems should be characterized differently than chaos in regular integer order systems.

It should be noted that the describing function approach usually requires at least  $-180^\circ$  of phase shift in the linear part of the feedback loop to ever predict Hopf bifurcations, and consequently chaos, for memoryless nonlinearities. Because the linear part can be a nonminimum phase transfer function, it is further conjectured that chaos can occur in systems with mathematical order less than three and probably less than one. Furthermore, the feedback configuration indicates that, as long as the linear part of the loop has at least  $-180^\circ$  of phase shift, the possibility of chaos in the system depends primarily on the nonlinearity and how its particular describing function behaves.

As has been demonstrated, the idea of fractional derivatives requires one to reconsider dynamic system concepts that are often taken for granted. Some of these concepts have been discussed in this report. Some others that require much further consideration are the concept of Lyapunov exponents for fractional states, the use of fractional states in which to embed attractors, and the relationship between fractional order and fractal dimension.

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TABLE I.—FRACTIONAL OPERATORS WITH APPROXIMATELY 2 db ERROR FROM  $w = 10^{-2}$  TO  $10^2$  rad/sec

$\frac{1}{s^{0.1}} \approx$	$\frac{220.4s^4 + 5004s^3 + 5038s^2 + 234.5s + 0.4840}{s^5 + 359.8s^4 + 5742s^3 + 4247s^2 + 147.7s + 0.2099}$
$\frac{1}{s^{0.2}} \approx$	$\frac{60.95s^4 + 816.9s^3 + 582.8s^2 + 23.24s + 0.04934}{s^5 + 134.0s^4 + 956.5s^3 + 383.5s^2 + 8.953s + 0.01821}$
$\frac{1}{s^{0.3}} \approx$	$\frac{23.76s^4 + 224.9s^3 + 129.1s^2 + 4.733s + 0.01052}{s^5 + 64.51s^4 + 252.2s^3 + 63.61s^2 + 1.104s + 0.002267}$
$\frac{1}{s^{0.4}} \approx$	$\frac{25.00s^4 + 558.5s^3 + 664.2s^2 + 44.15s + 0.1562}{s^5 + 125.6s^4 + 840.6s^3 + 317.2s^2 + 7.428s + 0.02343}$
$\frac{1}{s^{0.5}} \approx$	$\frac{15.97s^4 + 593.2s^3 + 1080s^2 + 135.4s + 1}{s^5 + 134.3s^4 + 1072s^3 + 543.4s^2 + 20.10s + 0.1259}$
$\frac{1}{s^{0.6}} \approx$	$\frac{8.579s^4 + 255.6s^3 + 405.3s^2 + 35.93s + 0.1696}{s^5 + 94.22s^4 + 472.9s^3 + 134.8s^2 + 2.639s + 0.009882}$
$\frac{1}{s^{0.7}} \approx$	$\frac{5.406s^4 + 177.6s^3 + 209.6s^2 + 9.197s + 0.01450}{s^5 + 88.12s^4 + 279.2s^3 + 33.30s^2 + 1.927s + 0.0002276}$
$\frac{1}{s^{0.8}} \approx$	$\frac{5.235s^3 + 1453s^2 + 5306s + 254.9}{s^4 + 658.1s^3 + 5700s^2 + 658.2s + 1}$
$\frac{1}{s^{0.9}} \approx$	$\frac{1.766s^2 + 38.27s + 4.914}{s^3 + 36.15s^2 + 7.789s + 0.01000}$

TABLE II.—LARGEST LYAPUNOV EXPONENTS FOUND IN THE STATE SPACE CONFIGURATION FOR  $q = 0.9, 1.0,$  AND  $1.1$  WHICH GIVES A TOTAL SYSTEM MATHEMATICAL ORDER OF 2.7, 3.0, AND 3.3, RESPECTIVELY

Mathematical system order	Order of system approximation	$\alpha$ -used	Exponents		
			$\lambda_1$	$\lambda_2$	$\lambda_3$
2.7	9	12.75	0.338	-0.000201	-0.132
3.0	3	9.50	0.248	-0.00412	-3.07
3.3	18	7.00	0.318	(a)	(a)

<sup>a</sup>These values were not calculated.

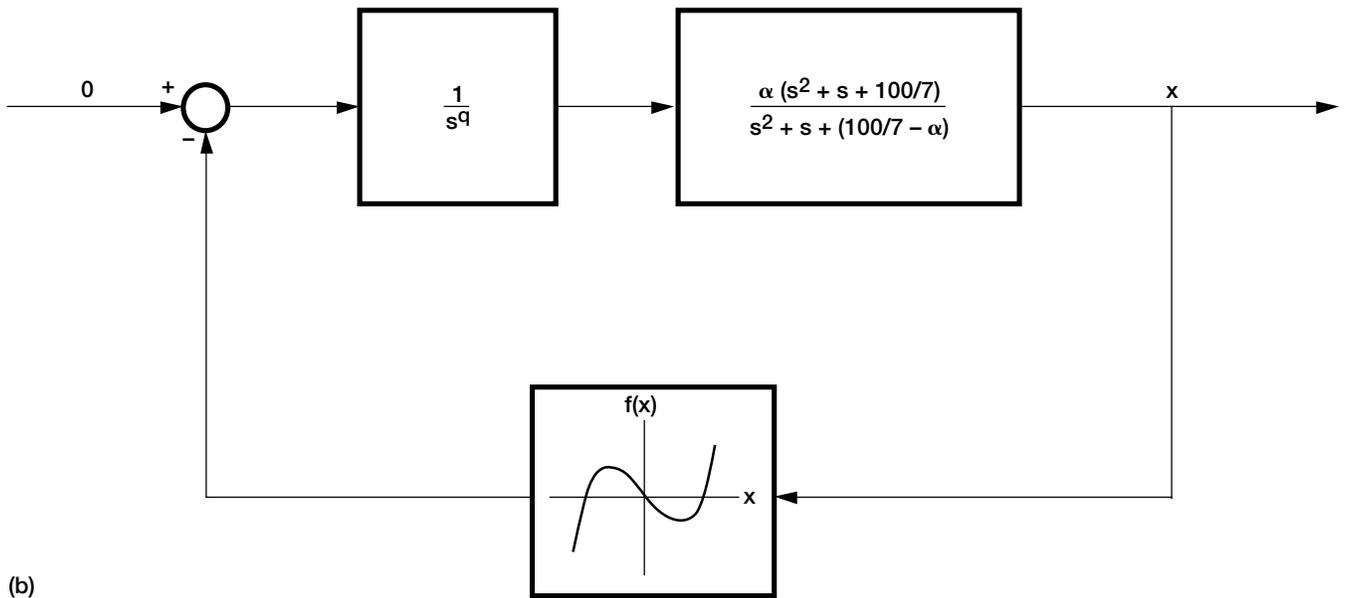
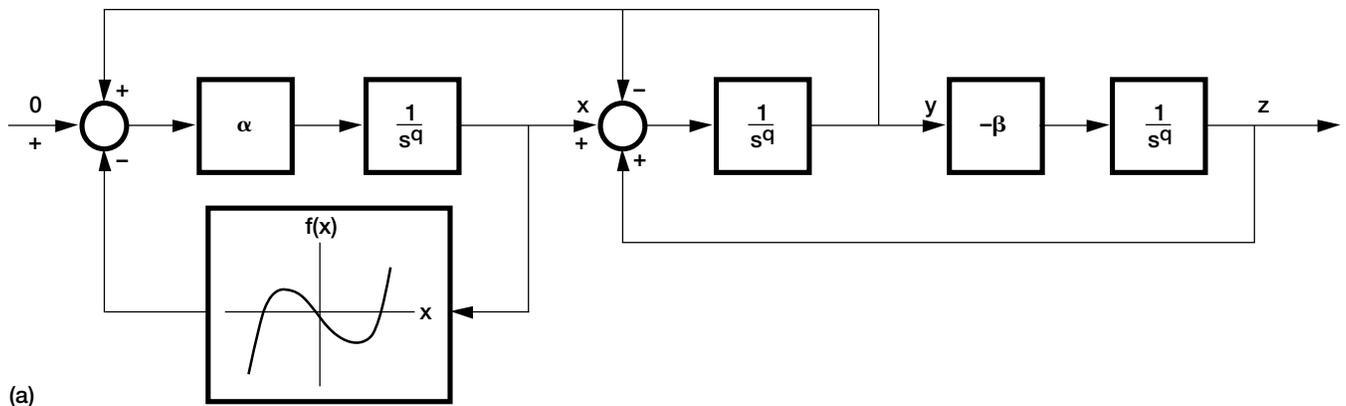


Figure 1.—System configurations for Chua's system which allow easy change of system order;  $q = 1$  is the nominal Chua system. (a) State space configuration. (b) Feedback configuration.

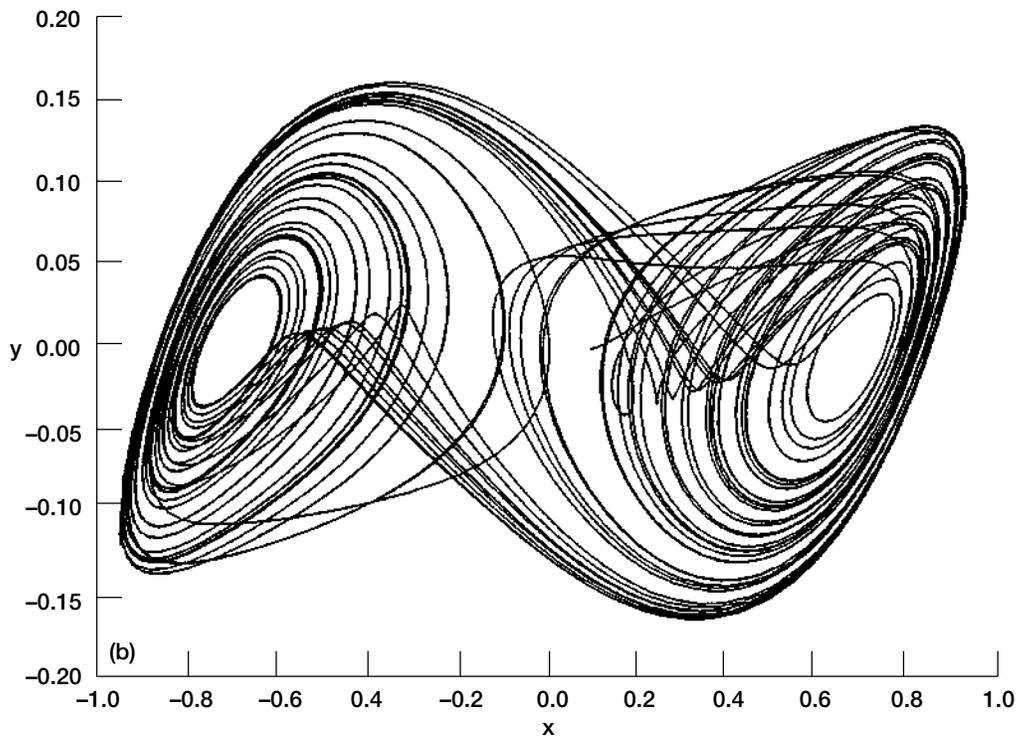
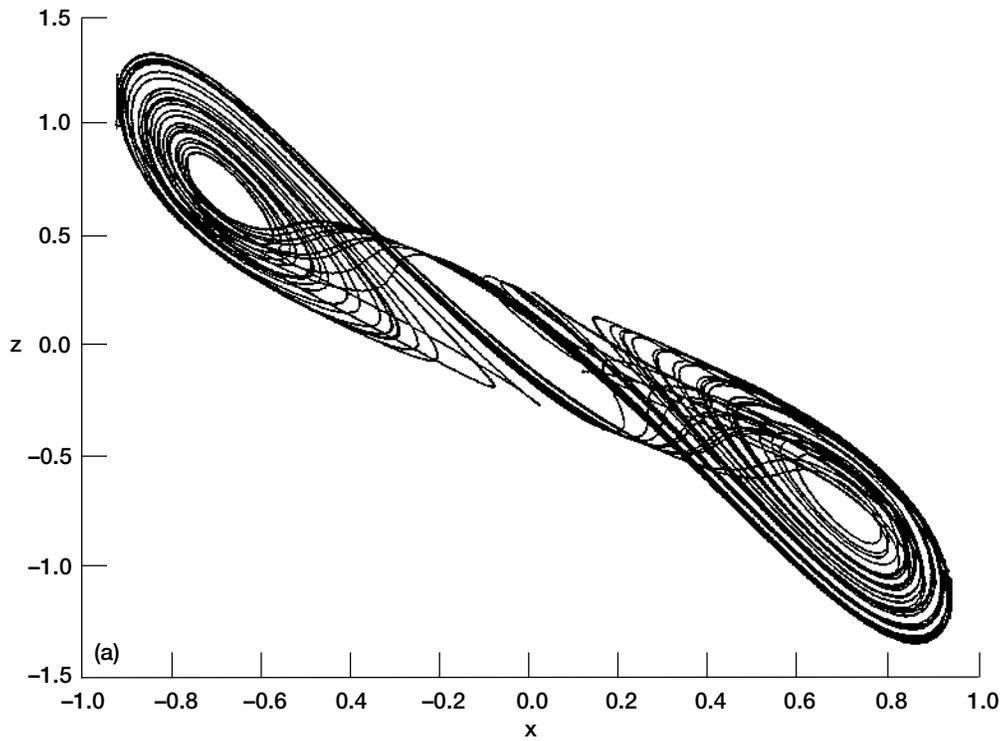


Figure 2.—Phase plane projections for the state space configuration of Chua's system. ( $t = 200, \Delta T = 0.05$ .) (a) Total mathematical system order is 3.0,  $\alpha = 9.5$ . (b) Total mathematical system order is 3.0,  $\alpha = 9.5$ .

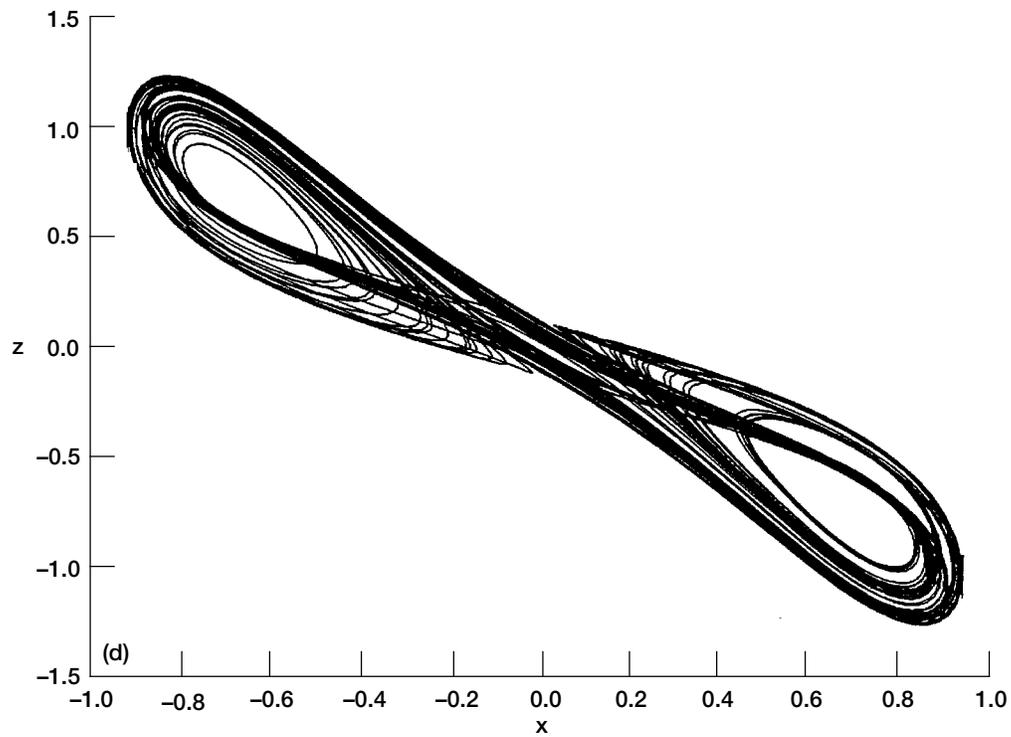
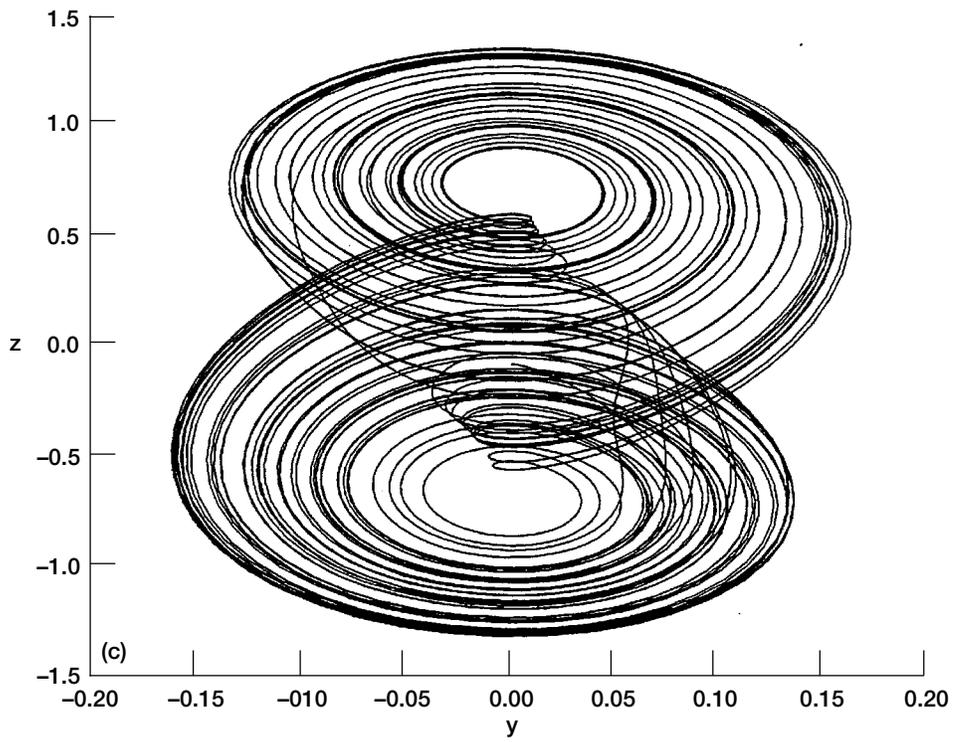


Figure 2.—Continued. Phase plane projections for the state space configuration of Chua's system. ( $t = 200, \Delta T = 0.05$ .) (c) Total mathematical system order is 3.0,  $\alpha = 9.5$ . (d) Total mathematical system order is 2.7,  $\alpha = 12.75$ .

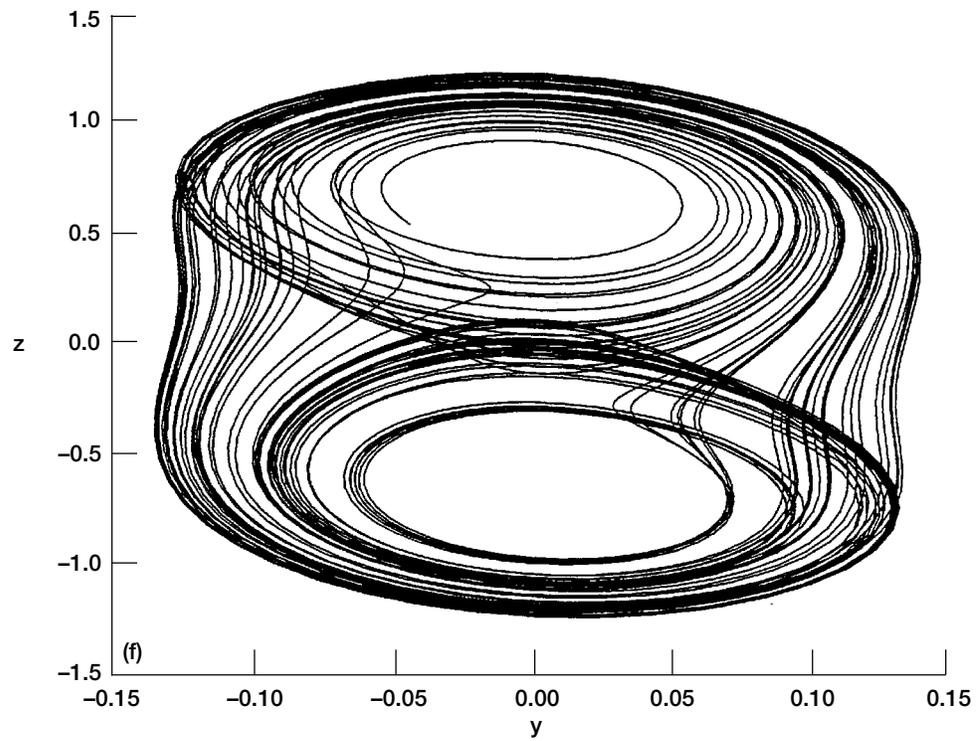
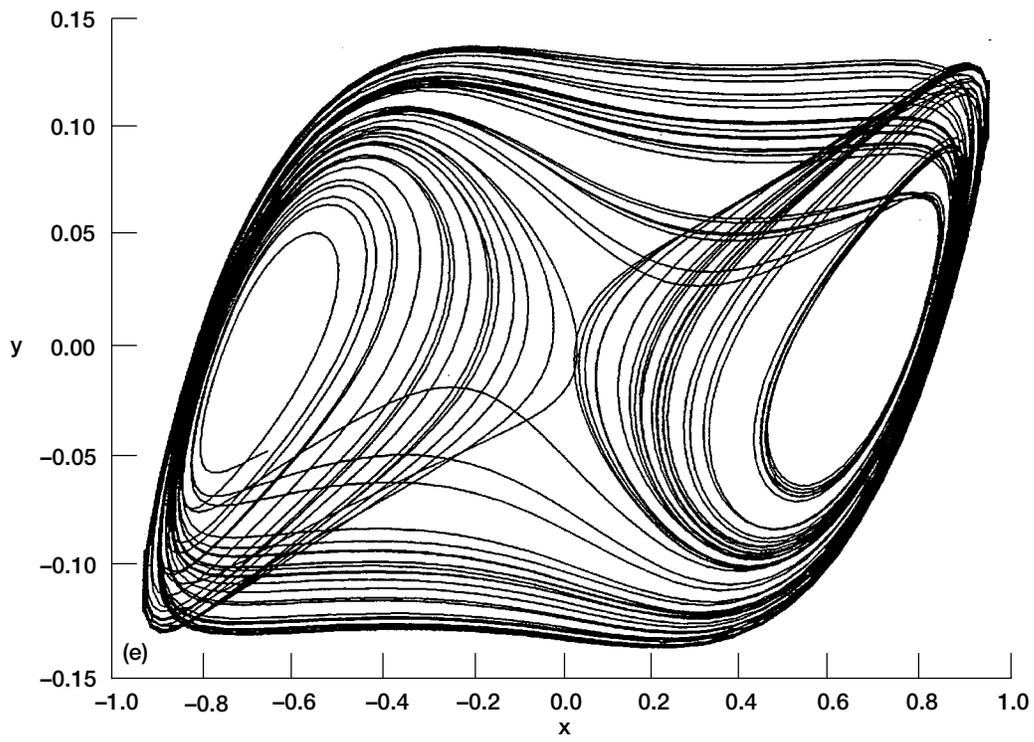


Figure 2.—Continued. Phase plane projections for the state space configuration of Chua's system. ( $t = 200$ ,  $\Delta T = 0.05$ .) (e) Total mathematical system order is 2.7,  $\alpha = 12.75$ . (f) Total mathematical system order is 2.7,  $\alpha = 12.75$ .

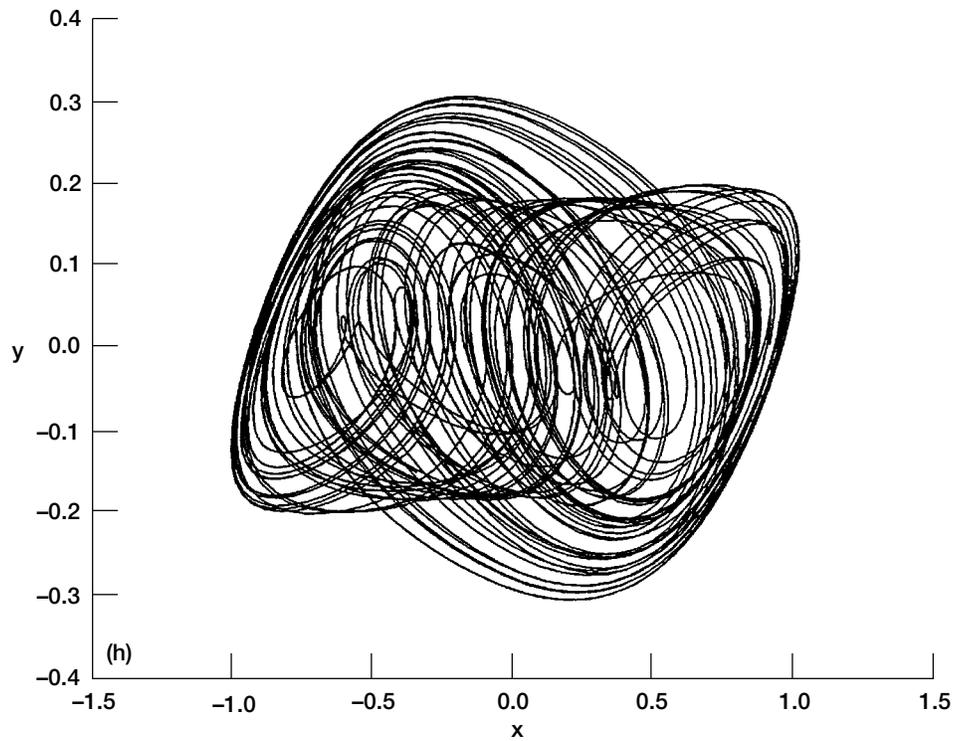
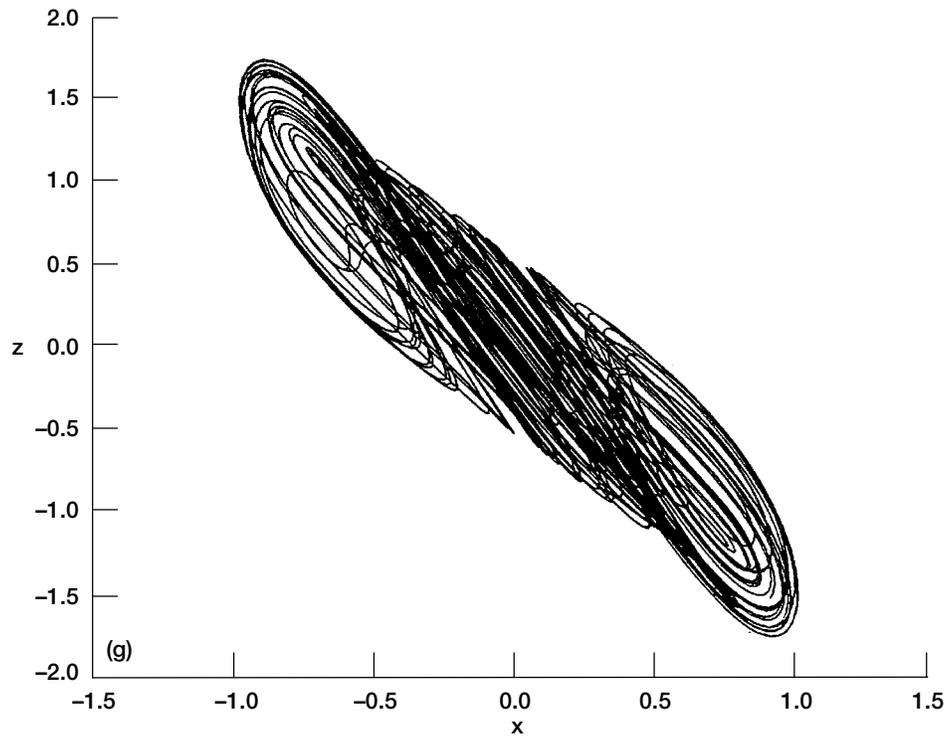


Figure 2.—Continued. Phase plane projections for the state space configuration of Chua's system. ( $t = 200$ ,  $\Delta T = 0.05$ .) (g) Total mathematical system order is 3.3,  $\alpha = 7.0$ . (h) Total mathematical system order is 3.3,  $\alpha = 7.0$ .

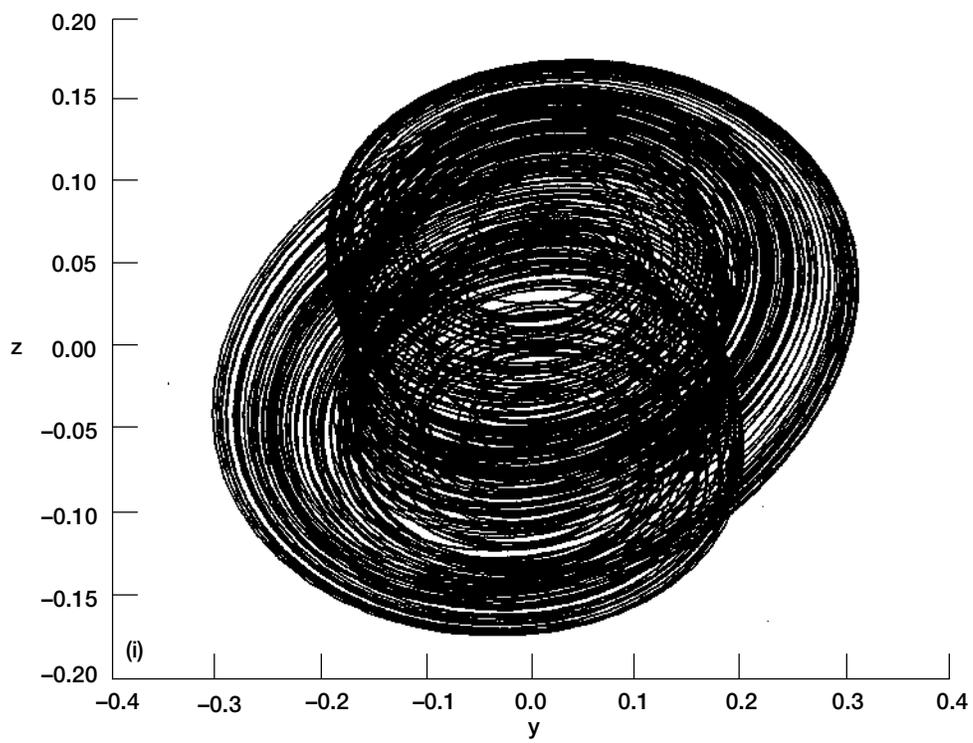


Figure 2.—Concluded. Phase plane projections for the state space configuration of Chua's system. ( $t = 200$ ,  $\Delta T = 0.05$ .) (i) Total mathematical system order is 3.3,  $\alpha = 7.0$ .

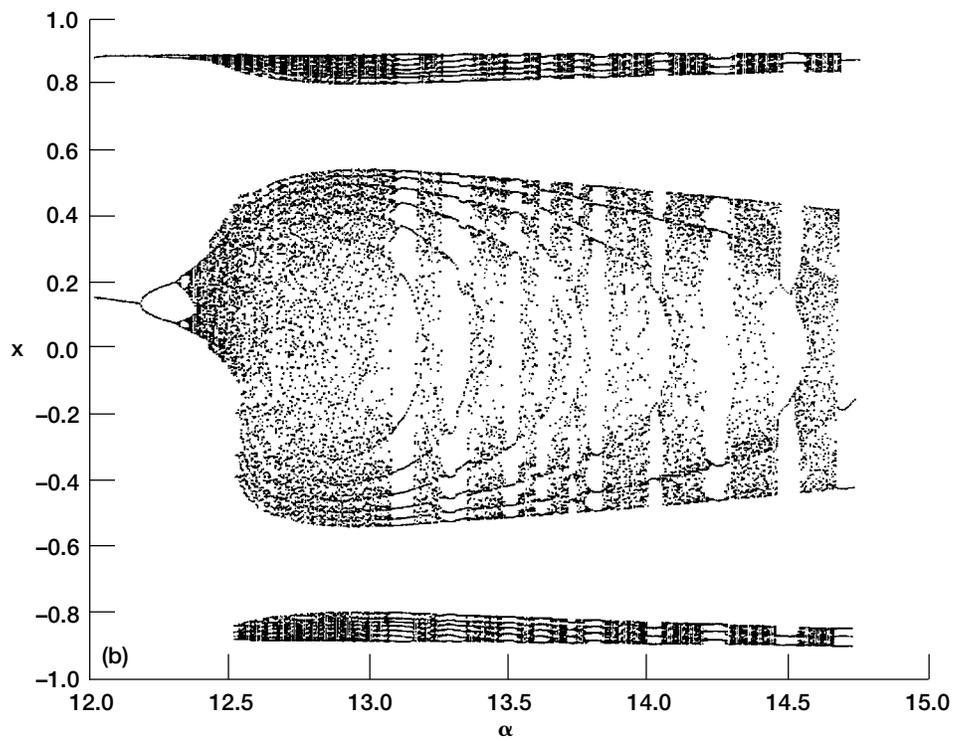
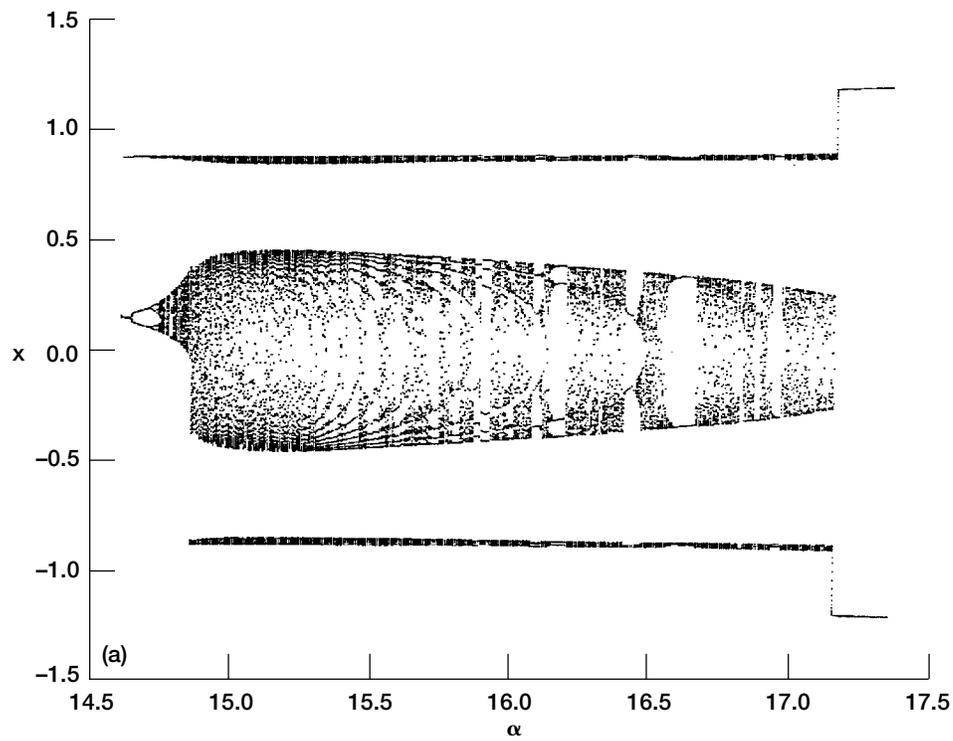


Figure 3.—Bifurcation diagram for the feedback configuration of Chua's system; maximum and minimum of  $x$  plotted against  $\alpha$ . (a) Fractional integral of order 0.6, total mathematical system order 2.6. (b) Fractional integral of order 0.7, total mathematical system order 2.7.

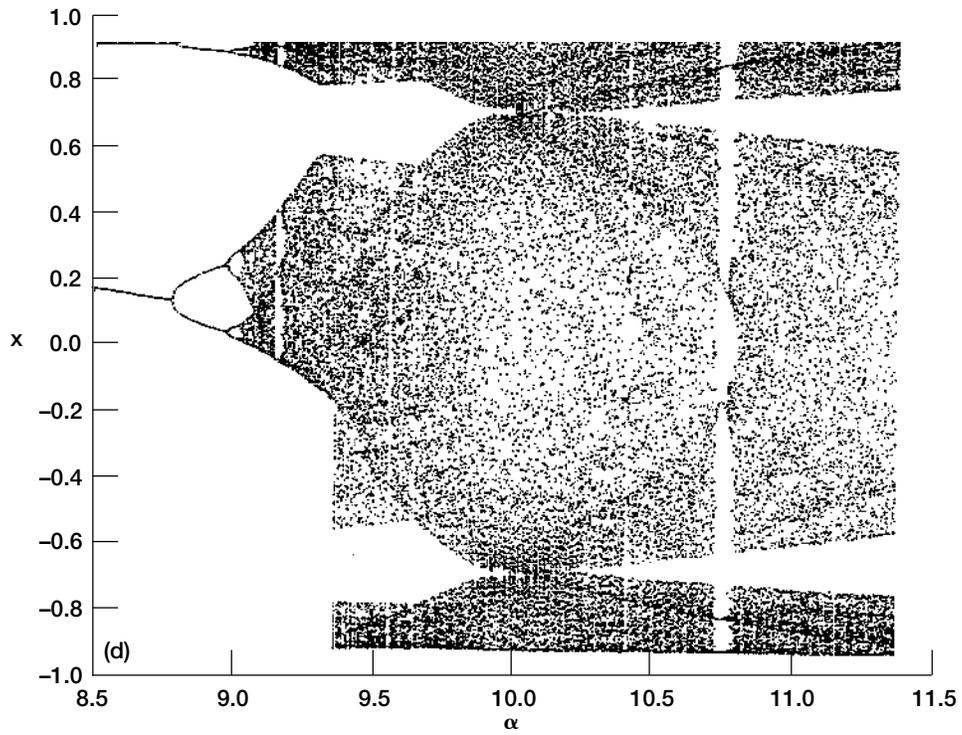
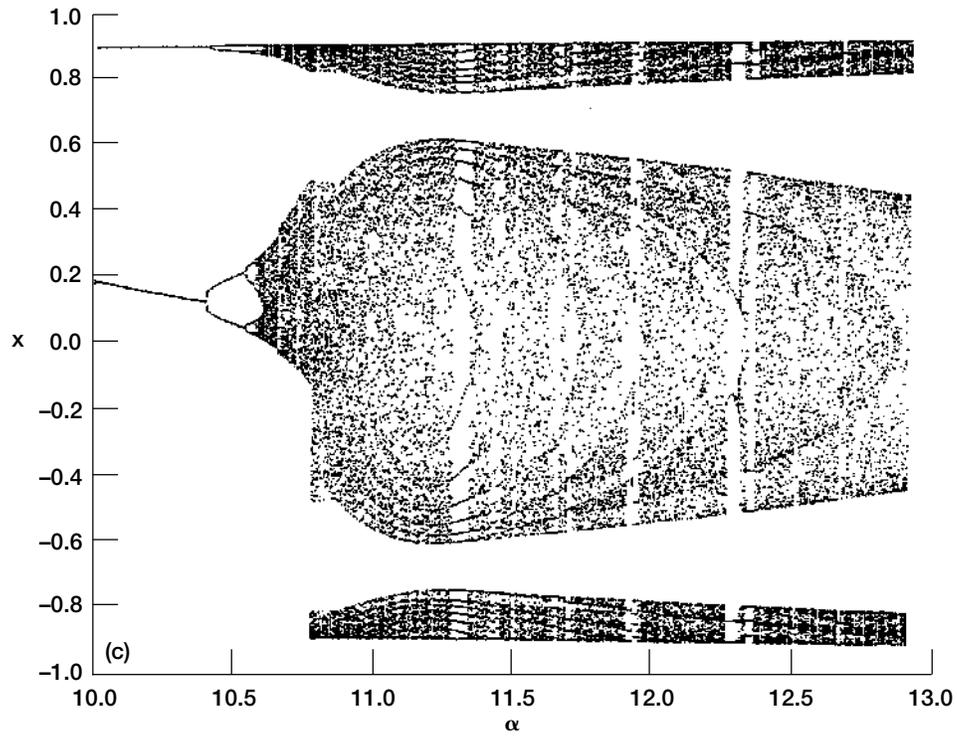


Figure 3.—Continued. Bifurcation diagram for the feedback configuration of Chua's system; maximum and minimum of  $x$  plotted against  $\alpha$ . (c) Fractional integral of order 0.8, total mathematical system order 2.8. (d) Fractional integral of order 0.9, total mathematical system order 2.9.

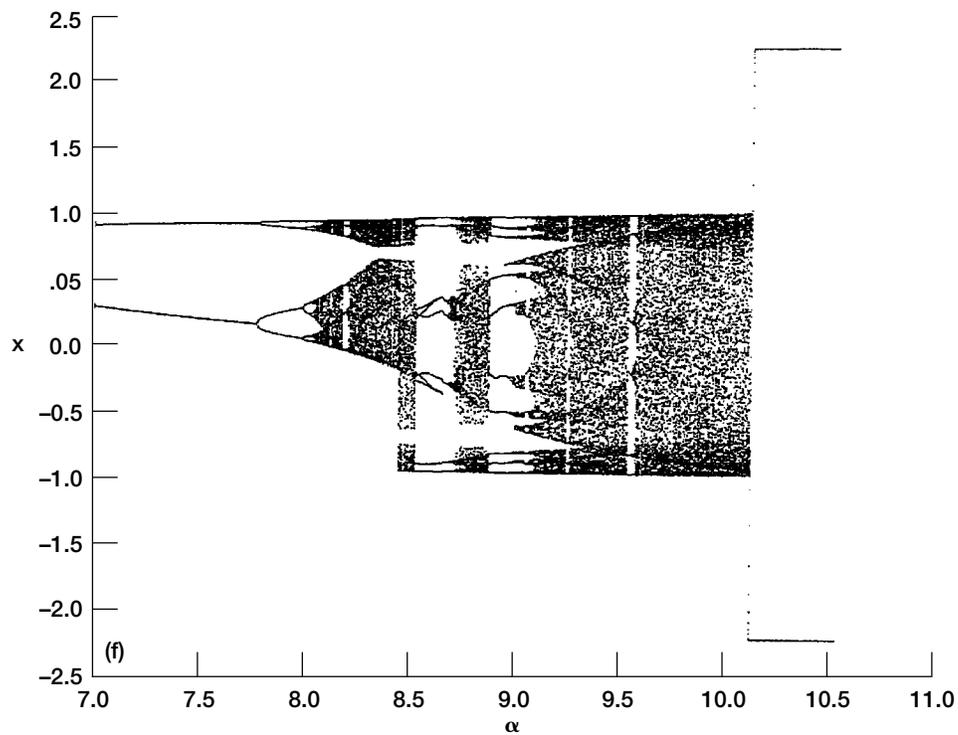
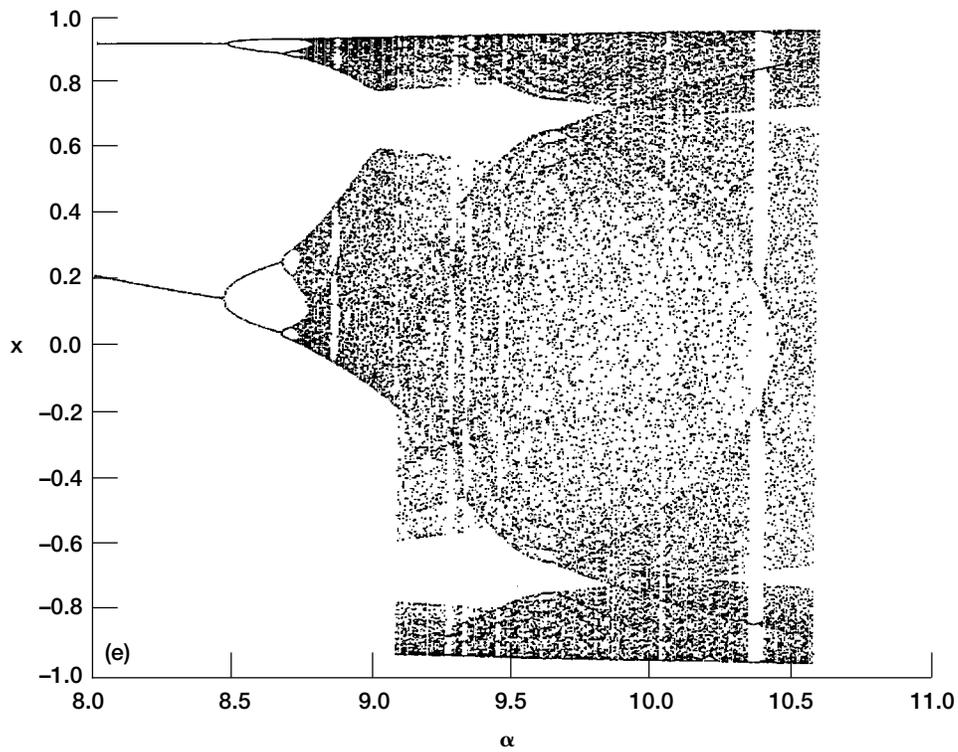


Figure 3.—Continued. Bifurcation diagram for the feedback configuration of Chua's system; maximum and minimum of  $x$  plotted against  $\alpha$ . (e) Fractional integral of order 1.0, total mathematical system order 3.0. (f) Fractional integral of order 1.1, total mathematical system order 3.1.

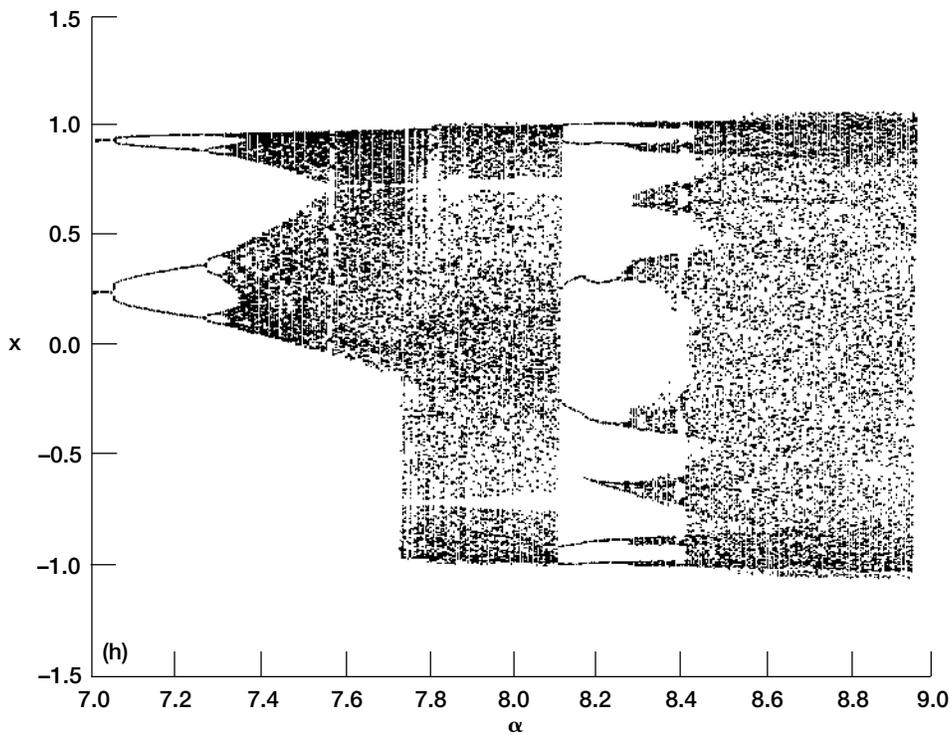
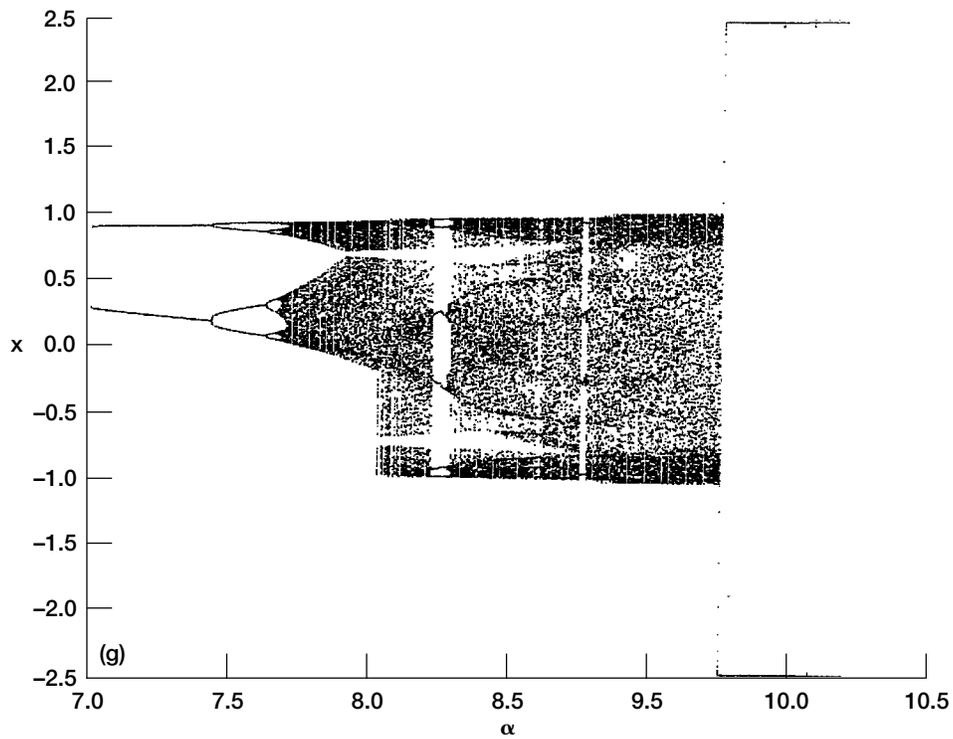


Figure 3.—Continued. Bifurcation diagram for the feedback configuration of Chua's system; maximum and minimum of  $x$  plotted against  $\alpha$ . (g) Fractional integral of order 1.2, total mathematical system order 3.2. (h) Fractional integral of order 3.1, total mathematical system order 3.3.

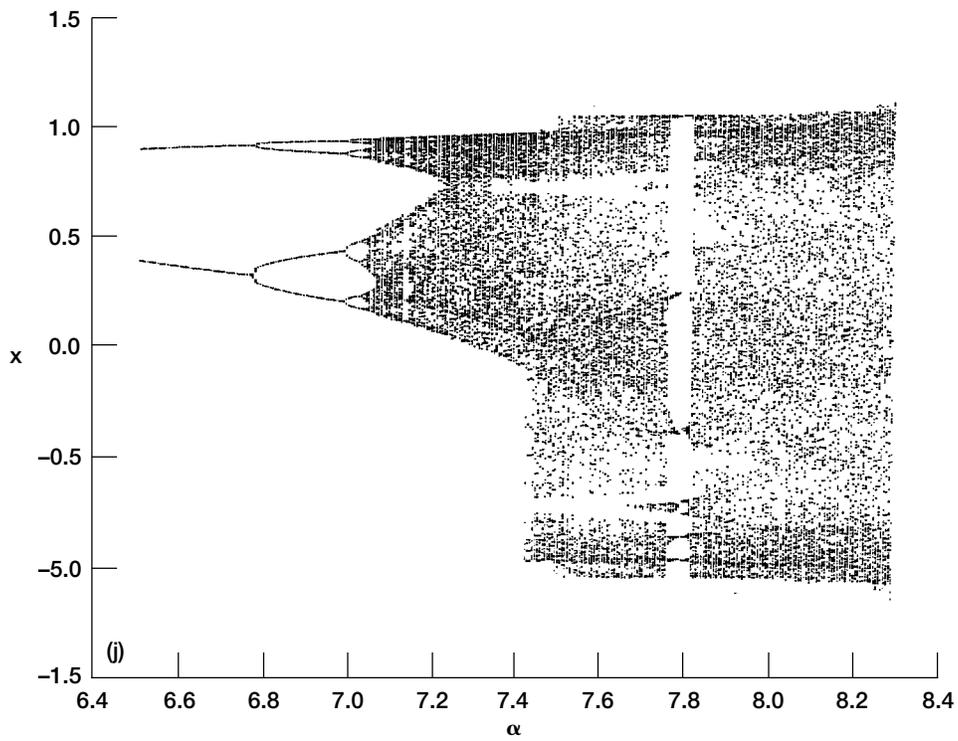
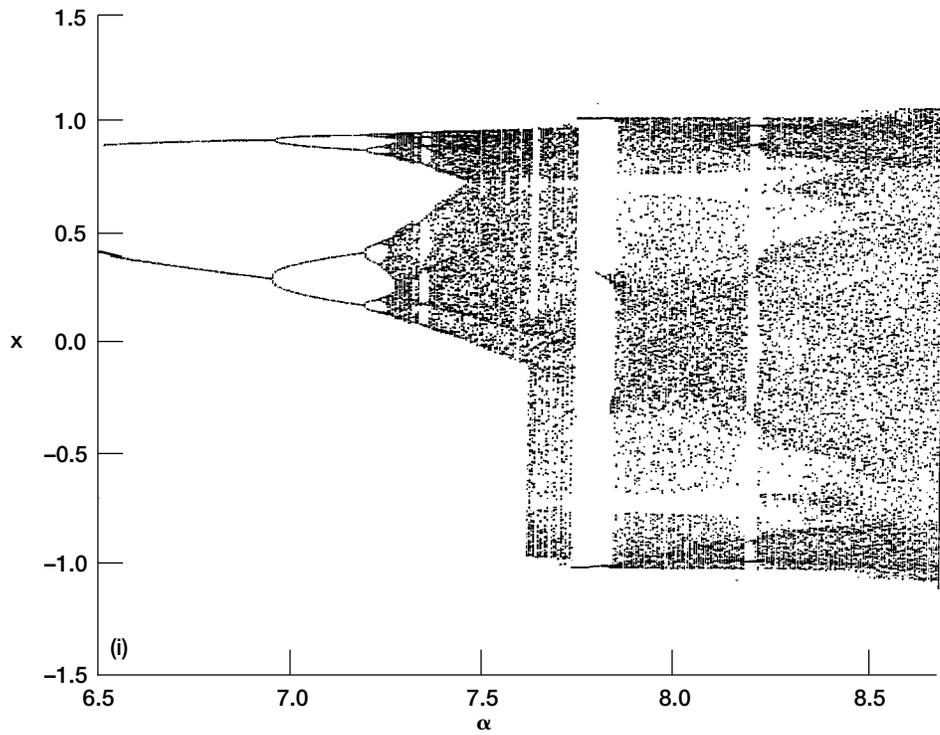


Figure 3.—Continued. Bifurcation diagram for the feedback configuration of Chua's system; maximum and minimum of  $x$  plotted against  $\alpha$ . (i) Fractional integral of order 1.4, total mathematical system order 3.4. (j) Fractional integral of order 1.5, total mathematical system order 3.5.

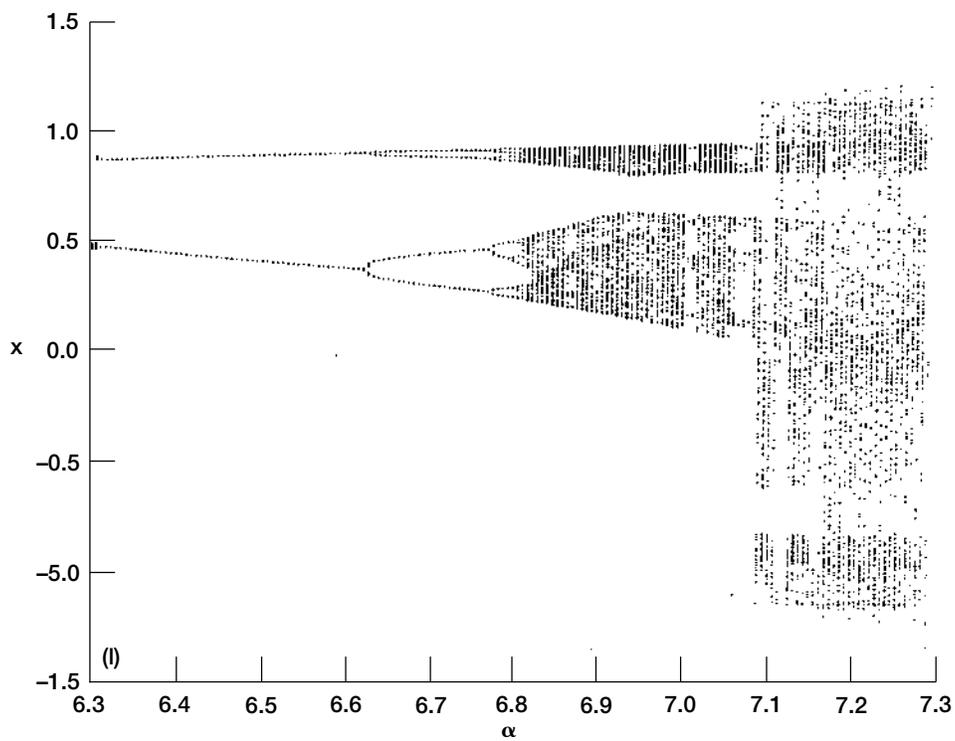
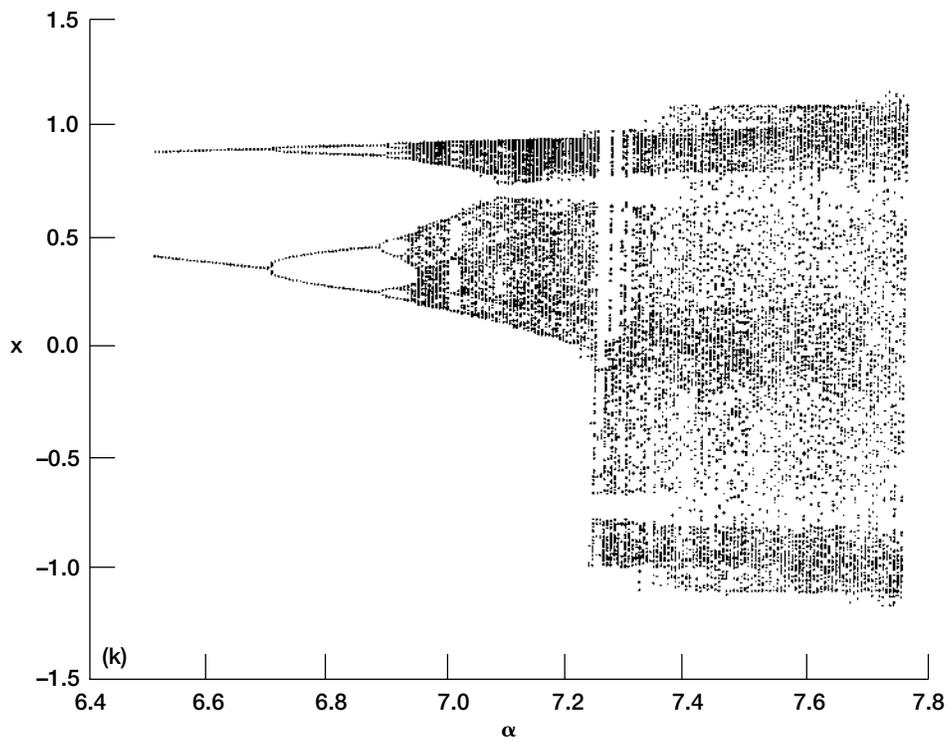


Figure 3.—Concluded. Bifurcation diagram for the feedback configuration of Chua's system; maximum and minimum of  $x$  plotted against  $\alpha$ . (k) Fractional integral of order 1.6, total mathematical system order 3.6. (l) Fractional integral of order 1.7, total mathematical system order 3.7.

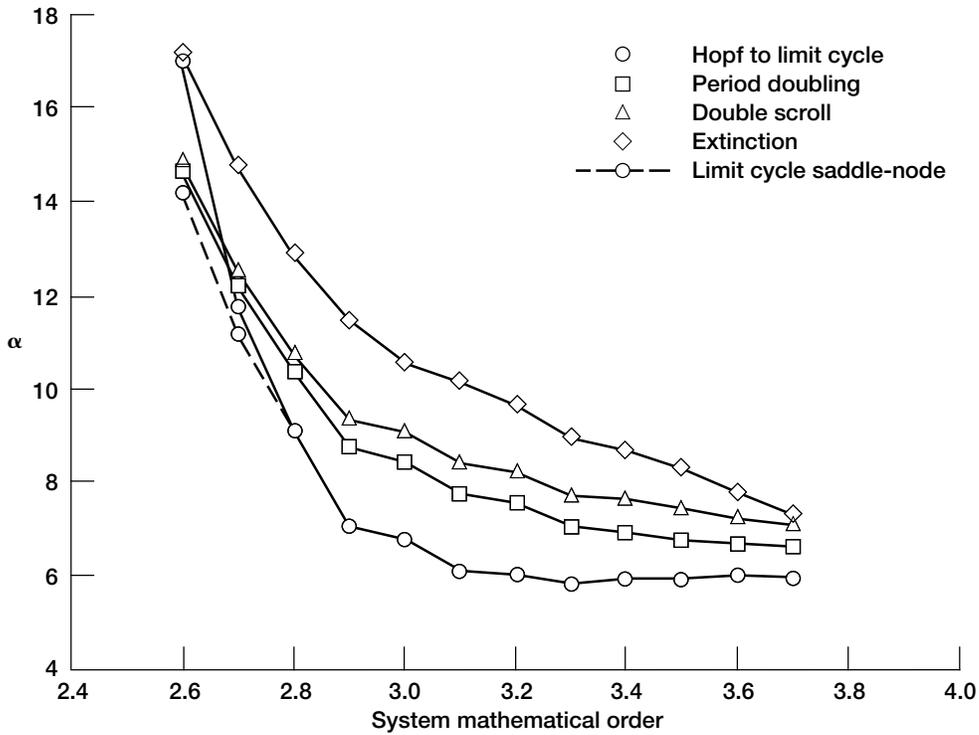


Figure 4.—Bifurcation diagram in the  $\alpha$  versus system mathematical order plane based on simulation studies of the fractional Chua system. Note that the saddle-node and sub-critical Hopf merge at 2.75.

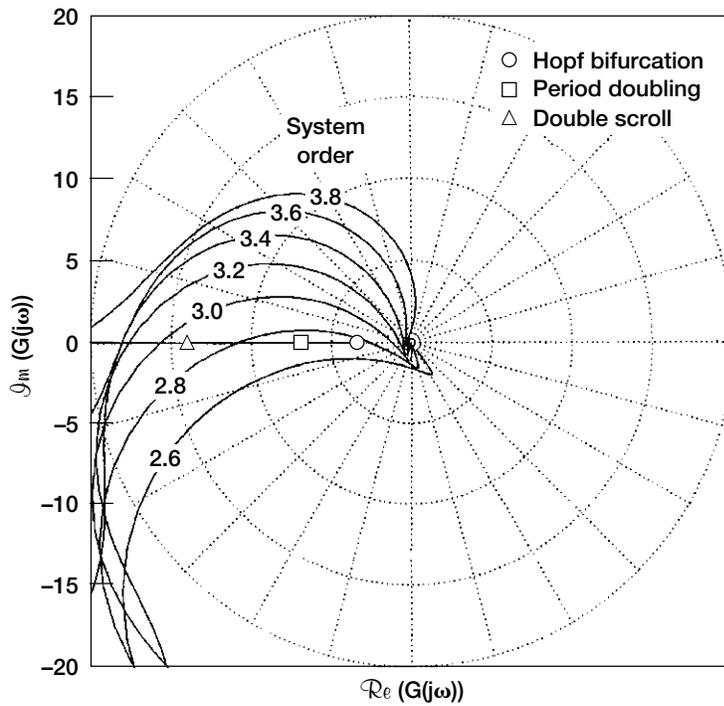


Figure 5.—Nyquist plane plot showing the frequency response of the linear part of figure 1 (set of curved lines) for various  $q$  and  $\alpha = 9.5$ ; and the describing function of the nonlinearity (solid line on real-axis from 3.5 to  $-20$  shown).

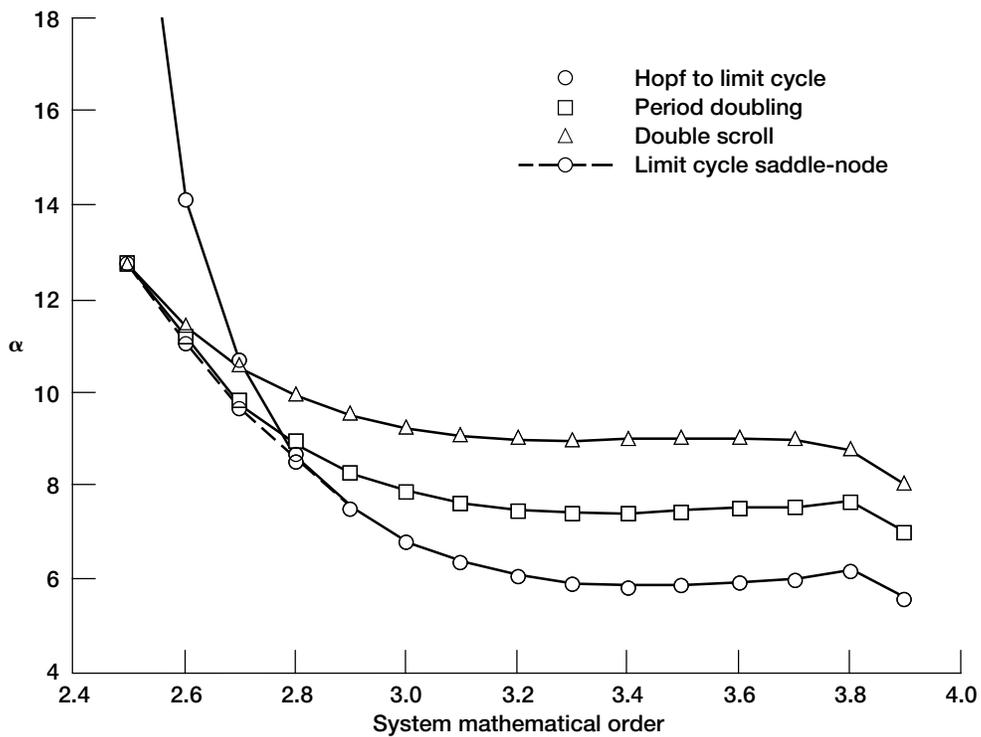


Figure 6.—Bifurcation diagram in the  $\alpha$  versus system mathematical order plane based on describing function analysis of the fractional Chua system. Note that the saddle-node and subcritical Hopf merge at 2.85.

# REPORT DOCUMENTATION PAGE

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